

Angle Trisection, Bhaskara's Proof, and Pythagorean Theorem

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Abstract

This paper deals with 1) angle trisection, 2) Bhaskara's first proof, and 3) Pythagorean theorem. The purpose of this paper is threefold. First, to show a new, direct method of trisecting the 90° angle using unmarked straight edge and compass; secondly, to show Bhaskara's first proof of the Pythagorean theorem ($c^2 = a^2 + b^2$) as embedded in this new, direct trisection of the 90° angle; lastly, to show the derivation of the Pythagorean theorem from this trisection of the 90° angle. This paper employs the direct dissection method. It concludes by presenting four points: a) the concept of trisectability as distinct from concept of constructability; b) the trisection of the 90° angle as really a new, different method; c) Bhaskara's first proof of the Pythagorean theorem as truly embedded in this trisection of the 90° angle and; d) another way of deriving Pythagorean theorem from this trisection of the 90° angle.

Keywords: right angle, angle trisection, Bhaskara's first proof, Pythagorean theorem

1.0 Introduction

In the history of mathematics, three famous problems in elementary geometry have occupied mathematicians since ancient Greek period. These are the quadrature of the circle, the duplication of the cube, and the trisection of an arbitrary angle. The Greeks set the rules for solving these problems. Only two tools are permitted to be used: **unmarked straight edge** and **compass**.

This paper focuses only on angle trisection. In layman's term, the problem of angle trisection may be stated as follows. To trisect a given arbitrary angle means to divide it into three equal angles.

This problem appears to be just simple at first glance, yet, in reality the general mathematical proof of whether or not angle trisection is possible has eluded mathematicians for centuries. In 1837 however, the French mathematician, Pierre Wantzel, finally gave the proof that angle trisection is generally impossible (Bailey, 2018).

The general proof of the impossibility of angle trisection means that there is no general method for trisecting just any given arbitrary angle. (We take note of the words *general* and *any*.) In spite of this general proof however, some angles can still be trisected using only the tools being permitted:

the unmarked straight edge and compass (Yates, 1971). The 45° and 90° angles for example, are known to be trisectable.

The 90° angle can be trisected using some methods. In this paper, we will show a different method for trisecting this angle. As far as the literature of this problem shows, this method is hitherto unknown in the history of this problem. This paper will first deal with this new method of trisecting the 90° angle.

Moreover, two new mathematical facts emerge from this new method. First, an inherent mathematical connection between Bhaskara's first proof of the Pythagorean theorem ($c^2 = a^2 + b^2$) and this trisection of the 90° angle exists. Bhaskara, an Indian mathematician in the 12th century A.D, gave geometrical proofs of the Pythagorean theorem (Loomis, 1968). His first proof is naturally embedded in this new method of trisecting the 90° angle; the proof would just naturally emerge from this trisection. Secondly, the Pythagorean theorem can be derived from this new method of trisecting the 90° angle.

With these new findings related to angle trisection, we are going to show the following three main points.

First, this paper will show the trisectability of the 90° angle into three equal angles using a new, direct method with the two tools being permitted: the unmarked straight edge and compass.

Secondly, this paper will show Bhaskara's first proof as naturally embedded in this trisection of the 90° angle. It will also show as well the inherent geometrical connection between Bhaskara's first proof and this new method of trisecting the 90° angle. Thus, this paper briefly presents Bhaskara's first proof.

Lastly, on the basis of Bhaskara's first proof,

the Pythagorean theorem may be said as a natural consequence in this new method of trisecting the 90° angle. However, this paper has to mathematically show the derivation of the Pythagorean theorem from this new method.

2.0 Methodology

This paper employs the direct dissection method. Here, the 90° angle itself is directly geometrically trisected or divided into three equal parts using the two tools, unmarked straight edge and compass. The two tools the ancient Greek mathematics permits for solving this problem would naturally lend more to the use of the direct dissection method. (This paper, however, would not touch anything of what may be called indirect method: i.e., the use of algebraic and/or trigonometric formulations in solving this geometrical problem.)

3.0 Results and Discussion

Constructibility and Trisectability Distinguished

This section distinguishes the words constructibility and trisectability. The importance of this distinction is that the latter illustrates the kind of method used in this paper.

In his article "Ram's Theorem for Trisection" Bhat (2019) says,

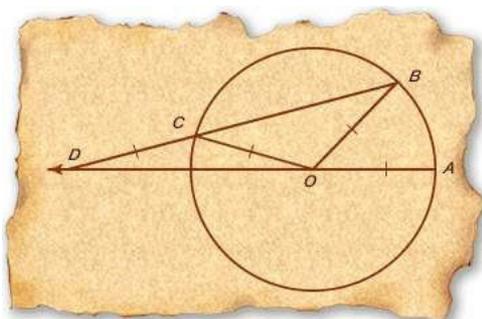
"it should also be noted that the proof of impossibility considers primarily the constructability of the angle of value equivalent to one-third of the given value and not the trisectability of the given angle directly."

The quoted passage differentiates the meanings of the words constructability and trisectability.

Constructibility, on the one hand, seems to refer to the idea whether or not an angle one-third of a given angle is constructible. Here, the problem of angle trisection is set first into an algebraic and/or trigonometric equation: $x^3 - 3x - 2^a = 0$, known as the trisection equation. This equation contains the following question: “Is it possible for all values of a , to find by a straightedge and compass construction a root x of the Trisection Equation?”

This paper would not deal with the trisection equation in itself here; it does not deal with the finding of the root x of this equation. Instead, this paper would merely point out the fact that the trisection equation is a sort of an indirect method or proof of proving the possibility or impossibility of angle trisection in general. It is indirect in the sense the problem of angle trisection is set first into algebraic formulation. In other words, it goes to algebra to see whether angle trisection in geometry is possible or not.

The concept of indirect method or proof can be illustrated using Archimedes’ method shown in Figure 1 (Heilbron, 2001). It does not start by algebraic formulation, yet it remains to be an indirect proof. The reason is as follows:



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Figure 1. Archimedes’ Method in Trisecting an Arbitrary Angle

In Fig. 1, the given arbitrary angle is $\angle BOA$. Archimedes constructed $\angle CDO$ | $\angle CDO = 1/3 \angle BOA$; Archimedes trisected the given angle that way. This method could still be considered indirect. The trisection of the given arbitrary $\angle BOA$ is done via the construction of $\angle CDO$; the given arbitrary angle $\angle BOA$ itself is not directly trisected. Thus, in this case, to say that an angle ($1/3$ of a given angle) is constructible is to say that an angle (one that is other than the given angle) can be constructed such as found in Archimedes’ method.

Trisectability, on the other hand, refers to the direct trisection of a given angle itself (as shown in the quoted passage above). A given angle itself is to be directly divided into three equal parts using unmarked straight edge and compass. To say that an angle is trisectable is to say that a given arbitrary angle is divisible into three equal angles **directly geometrically**.

Thus, the given arbitrary $\angle BOA$ in Figure 1 can be trisected directly geometrically by drawing two lines from point O, dividing $\angle BOA$ into three equal angles. This is direct method where the trisection is done directly on the given angle itself. No algebraic formulation of the problem is done.

The distinction between the words constructible and trisectable is now made clear. Again, this distinction is important here; for it illustrates the kind of method used in this paper. This paper uses the direct method or proof, or the concept of trisectability in the exposition of the subject matter.

Trisection of the 90° Angle

This section will show the trisection (trisectability) of the 90° angle, using the direct dissection method, with the unmarked straight edge and compass.

In Figure 2, let point D be the pivot point for the compass. We set the compass equal to the length of segment, DF. With point D as the pivot point, we make a circular arc from F to O, i.e., \widehat{FO} (the one below the diagonal line FO in Figure 2), that intersects with the \widehat{DT} . (Henceforth, the symbol on top of the two letters is used as symbol for arc.) We designate this point of intersection with letter H. We draw two lines; one from O to H and another from D to H. In this case, point H is the vertex of $\triangle DHO$.

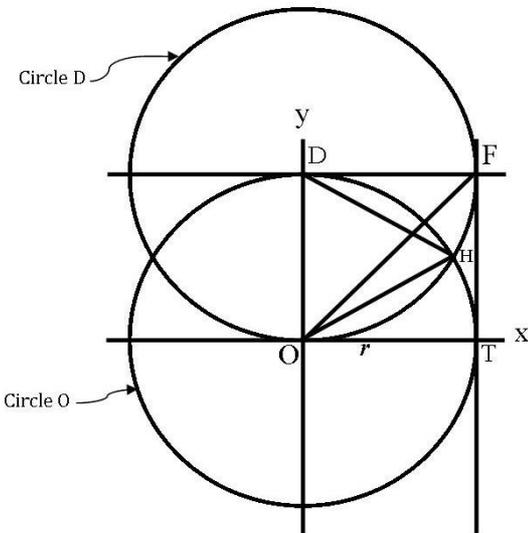


Figure 2. Intersection, H, of the \widehat{FO} (below diagonal line FO) and the \widehat{DT}

Next, in Figure 3, we do the same thing, but this time we let point T as the pivot point. We set the compass equal to the length of segment, TO. With point T as the pivot point, we make a circular arc from F to O, i.e., \widehat{FO} (see above the diagonal line FO), that intersects with the \widehat{DT} . We designate this point of intersection with letter L. We draw two lines; one from T to L and another from O to L. In this case, point L is the vertex of $\triangle OLT$.

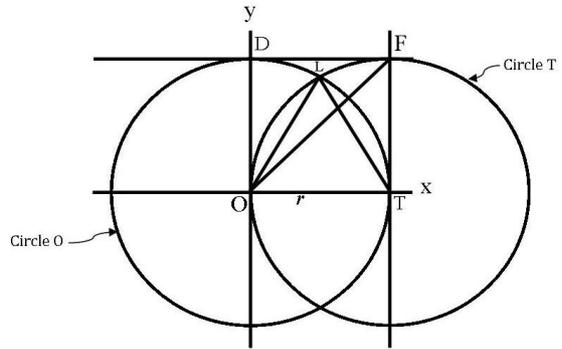


Figure 3. Intersection L of the \widehat{FO} (above diagonal line FO) and the \widehat{DT}

Now, from Figures 2 and 3, we have the following proposition:

Proposition 1

$$\triangle DHO = \triangle OLT = \text{equilateral triangle} \quad (1)$$

Proof:

In Figure 4, we let circle O, with its center at O, be the given circle. Its radius, $r = OT = OD$. We draw another circle having the same radius r , and let it be circle D with its center at D. Circles O and D intersect at point H, exactly midway between, or equidistant from the lines FD and TO. The point of intersection is located midway between points F and T, or is $\frac{1}{2}(FT)$ or $\frac{1}{2}(r)$ – although this point does not lie on the line, FT. It is because $FT = DO = r$.

Point H is on circle O and circle D. This point is equidistant from the centers of both circle O and circle D, respectively. With DH, OH, DO, and r we have...

$$DH = OH = DO = r \quad (2)$$

Therefore,

$$\triangle DHO = \text{equilateral triangle} \quad (3)$$

(see Figure 4 below)

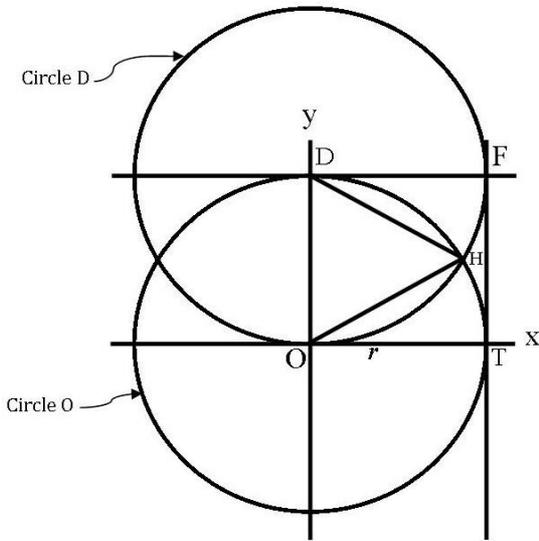


Figure 4. Circle O and Circle D with ΔDHO

In Figure 5, we still use the same circle O. We draw another circle T, with its center at T. They intersect at point L. By the same argument as in Figure 4, point L is equidistant to the centers of both circle O and circle T. With OL, TL, TO, and r we have...

$$OL = TL = TO = r \tag{4}$$

Therefore,

$$\Delta OLT = \text{equilateral triangle} \tag{5}$$

(see Figure 5 below)

Thus, proposition 1 is proven.

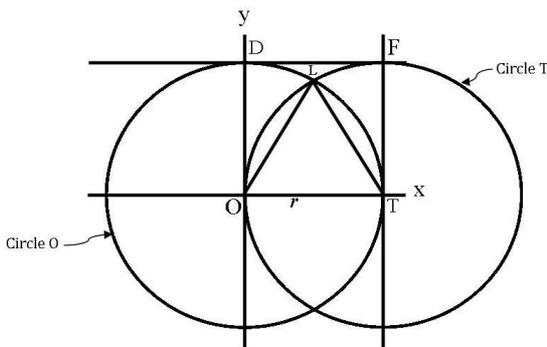


Figure 5. Circle O and Circle T with ΔOLT

Figure 6 shows the combination of the three circles, O, D, and T.

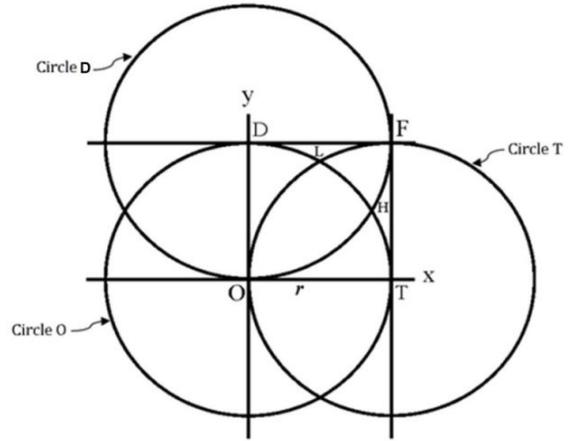


Figure 6. Circle O, Circle D, and Circle T

In Figure 7, ΔDHO and ΔOLT are combined with the circles D & T being removed.

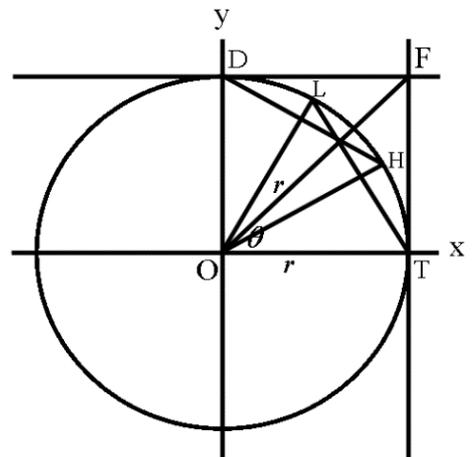


Figure 7. ΔDHO & ΔOLT

Now, $\Delta DHO = \Delta OLT$; they are proven to be equilateral triangles. Therefore,

$$\angle HDO = \angle DHO = \angle DOH = 60^\circ \tag{6}$$

And,

$$\angle LOT = \angle OLT = \angle OTL = 60^\circ \tag{7}$$

It follows that

$$\angle DOL = \angle LOH = \angle HOT = 30^\circ \quad (8)$$

This means that

$$\angle DOL + \angle LOH + \angle HOT = \angle DOT = 90^\circ \quad (9)$$

Thus, the right angle $\angle DOT$ is **trisected** in this way, using only the tools, unmarked straight edge and compass.

The Embedded Bhaskara’s First Proof of the Pythagorean Theorem

The Graphical Representation of Bhaskara’s First Proof

This subsection presents the graphical representation of Bhaskara’s first proof of the Pythagorean theorem. Bhaskara (1114–1185) offered two proofs of the Pythagorean theorem (Head, n.d.; Bogomolny, 2016). The image shown in Figure 8 is one of his two proofs.

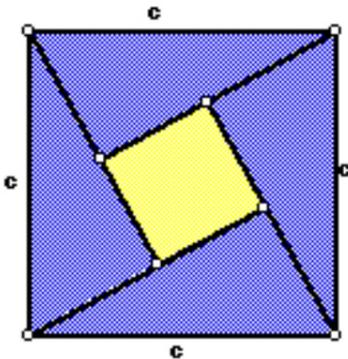


Figure 8. Bhaskara’s First Proof of the Pythagorean Theorem (<http://jwilson.coe.uga.edu/Pythagorean.html>, accessed 9-16-2019)

Bhaskara’s First Proof as Embedded in the Trisection of the 90° Angle

This subsection will show Bhaskara’s first proof of the Pythagorean theorem as naturally embedded in this new, direct geometrical trisection of the 90° angle. (The procedure in constructing Bhaskara’s

first proof, using unmarked straight edge and compass is shown in section 3.2 above.)

Figure 9 shows the construction of four circles in which Bhaskara’s first proof is embedded.

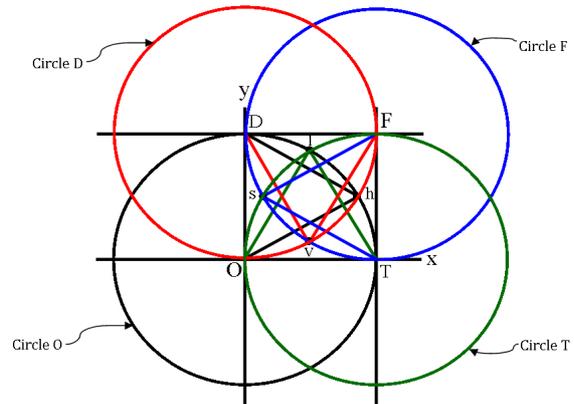


Figure 9. Bhaskara’s First Proof Embedded in the Four Circles

In Figure 9, there are four circles: O, T, F, and D. Let us focus on the quarter square $\square DOTF$ and the four circles in the first quadrant of the Cartesian coordinates x & y .

First is circle O (colored black). \widehat{DT} (colored black) connects points D and T. \widehat{DT} (black) also intersects with \widehat{FO} (red). We designate their point of intersection with lower case letter h. We draw the lines, hD and hO. With line OD, we have the *equilateral* $\triangle DhO$.

Second is circle T (colored green). \widehat{FO} (colored green) connects points F and O. \widehat{FO} (green) also intersects with \widehat{DT} (black). We designate their point of intersection with lower case letter l. We draw the lines, lO and lT. With line OT, we have the *equilateral* $\triangle lTlO$.

Third is circle D (colored red). \widehat{FO} (colored red) connects points F and O. \widehat{FO} (colored red) also intersects with \widehat{DT} (colored blue). We designate their point of intersection with lower case letter v.

We draw the lines vD and vF . With line OD , we have the *equilateral* ΔDvF .

Fourth is circle F (colored blue). \widehat{DT} (colored blue) connects points D and T . \widehat{DT} (colored blue) also intersects with \widehat{FO} (colored green). We designate their point of intersection with lower case letter s . We draw lines sT and sF . With line FT , we have the *equilateral* ΔFsT .

Now, ΔDhO , ΔTIO , ΔDvF , and ΔFsT are equilateral triangles. It follows that...

$$\Delta DhO = \Delta TIO = \Delta DvF = \Delta FsT \quad (10)$$

In Figure 10, we remove the three circles, D , F , and T , leaving only circle O (colored black).

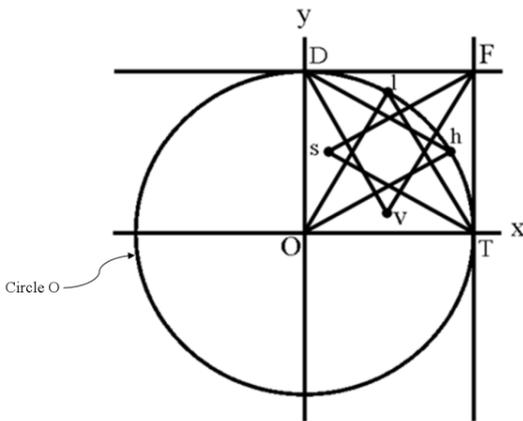


Figure 10. Circle O with the Four Equilateral Triangles

The intersection of the equilateral triangles, ΔDhO , ΔTIO , ΔDvF , and ΔFsT in Figure 10 looks like a star. We now remove circle O , as shown in Figure 11.

Bhaskara's first proof of the Pythagorean theorem is not yet clear and distinct in Figure 11. The legs of the four equilateral triangles cross each other.

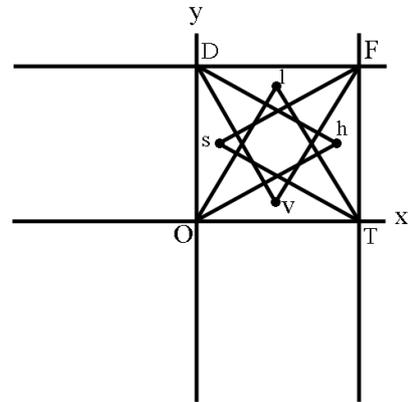


Figure 11. The Four Equilateral Triangles

We now remove either the right leg or left leg of each of the four equilateral triangles. See Figures 12 and 13.

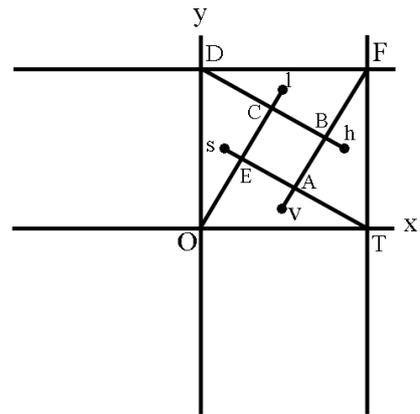


Figure 12. Right Legs of the Four Equilateral Triangles Removed

The right legs Oh , Tl , Fs , and Dv are removed in Figure 12. The four triangles ΔOCD , ΔTEO , ΔFAT , and ΔDBF are now clear. The four capital letters A , B , C & E are used to designate the angles: $\angle FAT$, $\angle DBF$, $\angle OCD$, and $\angle TEO$. At this point, Bhaskara's first proof emerges.

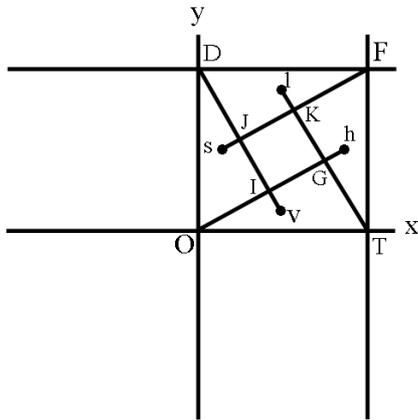


Figure 13. Left Legs of the Four Equilateral Triangles Removed

The left legs OI, Dh, Fv, and Ts are removed in Figure 13. The four triangles ΔOGT , ΔTKF , ΔFJD , and ΔDIO are now clear. The four capital letters G, K, J, and I are used to designate the angles: $\angle OGT$, $\angle DIO$, $\angle FJD$, and $\angle TKF$. Similarly, at this point, Bhaskara’s first proof emerges.

What remains to be shown now is whether $\angle FAT$, $\angle DBF$, $\angle OCD$, $\angle TEO$ (in Figure 12), $\angle OGT$, $\angle DIO$, $\angle FJD$, and $\angle TKF$ (in Figure 13) are all right angles.

Theorem

Angles A, B, C, E, G, I, J, & K are all right angles.

Proof

In Figure 12,

$$\therefore \angle sTO = \angle lOD = \angle hDF = \angle vFT = 30^\circ \quad (11)$$

And,

$$\therefore \angle FTA = \angle TOE = \angle ODC = \angle DFB = 60^\circ \quad (12)$$

$$\therefore \angle FAT = \angle DBF = \angle OCD = \angle TEO = 90^\circ \quad (13)$$

In Figure 13,

$$\therefore \angle sFD = \angle lTF = \angle hOT = \angle vDO = 30^\circ \quad (14)$$

And

$$\therefore \angle FDJ = \angle TFK = \angle OTG = \angle DOI = 60^\circ \quad (15)$$

$$\therefore \angle OGT = \angle DIO = \angle FJD = \angle TKF = 90^\circ \quad (16)$$

Figures 12 and 13 show Bhaskara’s first proof of

the Pythagorean theorem as naturally embedded. Figures 12 and 13, trimmed and the lowercase letters inside the square removed are shown in Figures 14 and 15 below, respectively.

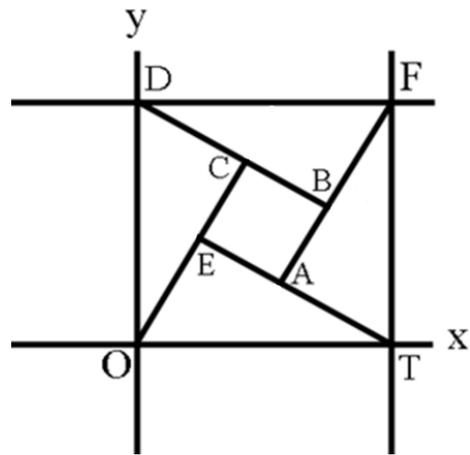


Figure 14. Bhaskara’s First Proof with $\angle FAT$, $\angle DBF$, $\angle OCD$, $\angle TEO$

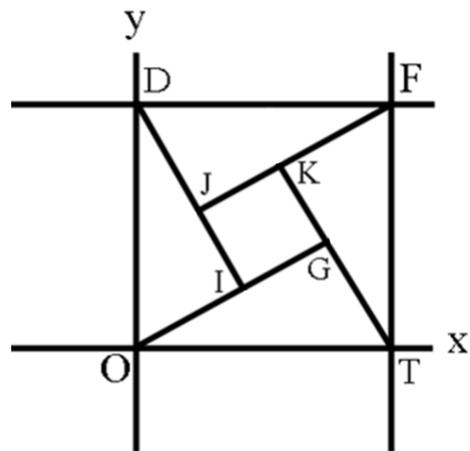


Figure 15. Bhaskara’s First Proof with $\angle OGT$, $\angle DIO$, $\angle FJD$, $\angle TKF$

Bhaskara’s first proof of the Pythagorean theorem is now clear as indicated by the four right triangles and the small square. The four right

triangles ΔOCD , ΔTEO , ΔFAT , ΔDBF and the small square $\square ECBA$ have now clearly emerged in Figure 14. Similarly, the right triangles ΔOGT , ΔTKF , ΔFJD , ΔDIO and the small square $\square IJKG$ have now clearly emerged in Figure 15.

Derivation of the Pythagorean Theorem

This section will show the derivation of the Pythagorean theorem (Zimba, 2009) using this new, direct trisection of the 90° angle together with Bhaskara’s first proof.

In Figure 16, we let $c = FT$, $a = FA$, and $b = TA$.

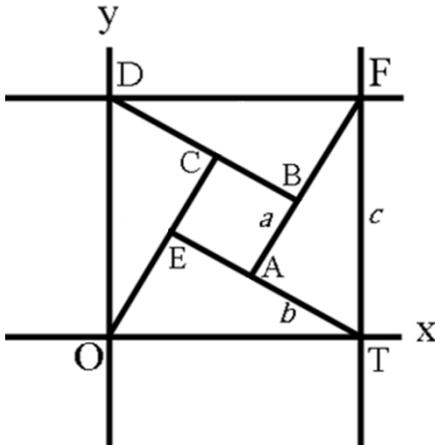


Figure 16. $\angle FAT$ with Sides a , b , & c

Then, we have

$$a = c \sin 60 = c \frac{\sqrt{3}}{2} \tag{17}$$

$$b = c \cos 60 = c \frac{1}{2} \text{ (here, } b = \frac{1}{2} r \text{)} \tag{18}$$

Let A_s be the area of the whole $\square DOTF$ in Figure 16. Then, we have,

$$A_s = c^2 \tag{19}$$

Let A_t be the area of the four triangles. Then, we have

$$A_t = \Delta OCD + \Delta TEO + \Delta FAT + \Delta DBF \tag{20}$$

(We may use only one triangle and multiply it by 4, since $\Delta FAT = \Delta DBF = \Delta OCD = \Delta TEO$)

Then, we have

$$A_t = 4 \left(\frac{1}{2} ab \right) \tag{21}$$

$$A_t = 2ab \tag{22}$$

Substituting 17 and 18 on 22, we have

$$A_t = 2 \left(c \frac{\sqrt{3}}{2} \right) \left(c \frac{1}{2} \right) \tag{23}$$

Multiplying $\left(c \frac{\sqrt{3}}{2} \right) \left(c \frac{1}{2} \right)$, we have

$$A_t = 2 c^2 \frac{\sqrt{3}}{4} \tag{24}$$

$$A_t = c^2 \frac{\sqrt{3}}{2} \tag{25}$$

Since $\frac{\sqrt{3}}{2} = \sin 60$, then we have

$$A_t = c^2 \sin 60 \tag{26}$$

Let A_{ss} be the area of the small square $\square CEAB$ in Figure 16. Then, we have

$$A_{ss} = (a - b)^2 \tag{27}$$

Substituting 17 and 18 on 27, we have

$$A_{ss} = \left(c \frac{\sqrt{3}}{2} - c \frac{1}{2} \right)^2 \tag{28}$$

$$A_{ss} = \left(c \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) \right)^2 \tag{29}$$

$$A_{ss} = c^2 \left(\frac{\sqrt{3}-1}{2} \right)^2 \tag{30}$$

$$A_{ss} = c^2 \left(\frac{3-2\sqrt{3}+1}{4} \right) \tag{31}$$

$$A_{ss} = c^2 \left(\frac{3}{4} + \frac{1}{4} - \frac{\sqrt{3}}{2} \right) \tag{32}$$

$$A_{ss} = c^2 \left(1 - \frac{\sqrt{3}}{2} \right) \tag{33}$$

Since $\frac{\sqrt{3}}{2} = \sin 60$, then we have

$$A_{ss} = c^2 (1 - \sin 60) \tag{34}$$

The Area of the bigger square, $A_s = c^2$, is equal to the total area, A_{total} , of the four triangles, $2ab$, plus the small square, $(a - b)^2$. Thus, we have

$$A_{total} = A_s = A_t + A_{ss} \tag{35}$$

Since $A_s = c^2$, then we have

$$c^2 = 2ab + (a - b)^2 \tag{36}$$

Substituting 17 and 18 on 36, we have

$$c^2 = \left(c^2 \frac{\sqrt{3}}{2} \right) + \left(c^2 \left(1 - \frac{\sqrt{3}}{2} \right) \right) \tag{37}$$

$$c^2 = c^2 \left(\left(\frac{\sqrt{3}}{2} \right) + \left(1 - \frac{\sqrt{3}}{2} \right) \right) \quad (38)$$

$$c^2 = c^2 \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 1 \right) \quad (39)$$

$$c^2 = c^2 \quad (40)$$

Or,

$$c^2 = a^2 + b^2 \quad (41)$$

Substituting 17 and 18 on 41, we have

$$c^2 = \left(c \frac{\sqrt{3}}{2} \right)^2 + \left(c \frac{1}{2} \right)^2 \quad (42)$$

$$c^2 = \frac{3c^2}{4} + \frac{c^2}{4} \quad (43)$$

$$c^2 = \frac{3c^2 + c^2}{4} \quad (44)$$

$$c^2 = \frac{4c^2}{4} \quad (45)$$

$$c^2 = c^2 \quad (46)$$

Therefore,

$$c^2 = a^2 + b^2 \quad (47)$$

Thus, the Pythagorean theorem is derived from this new, direct trisection of the 90° angle together with Bhaskara's first proof of the Pythagorean theorem.

4.0 Conclusion

There are four important points to be mentioned here.

Firstly, we conclude that the **trisection** of an angle is not totally impossible in terms of the concept of **trisectability** (though in terms of the concept of **constructability**, angle trisection is said to be generally impossible). The important distinction between **trisectability** and **constructability** is made clear.

Secondly, although it is not new that the 90° angle is trisectable, we conclude that it is geometrically trisectable using a new **hitherto unknown** way or method by means of the tools

being permitted: the **unmarked straight edge** and **compass**.

Thirdly, we conclude that Bhaskara's first proof of the Pythagorean theorem is being truly **embedded** in this new, direct trisection of the 90° angle using **unmarked straight edge** and **compass**.

Lastly, we conclude that the Pythagorean theorem can be derived from this new, direct trisection of the 90° angle in which Bhaskara's first proof is embedded.

Some of the facts presented in this short paper are something new. The method of trisecting the 90° angle shown in Figure 9 is entirely new, as far as the literature of this antique problem in elementary geometry shows.

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