On a Fractal Representation of the Density of Primes

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Abstract

The number of primes less or equal to a real number x, π(x), has been approximated in the past by the reciprocal of the logarithm of the number x. Such an approximation works well when x is large but it can be poor when x is small. This paper introduces a fractal formalism to provide more flexible approximation to the density of primes less or equal to a number x using the λ(s)-fractal spectrum. Results revealed that the density of primes less than or equal to x can be modeled as a monofractal probability mass function with high fractal dimension for large x. High fractal dimensions can often be decomposed to form a multifractal representation. The fractal density representation of the density of primes is closely linked to the Riemann zeta function and, thus, to the famous unsolved Riemann hypothesis.

Keywords: fractal density, prime number theorem, Riemann hypothesis

1.0 Introduction

Legendre was first to conjecture on the number of primes $(\pi(x))$, less than or equal to a positive real number x in 1798. He stated:

$$(1)\ldots\pi(x)\sim\frac{x}{\log x-1.08366}$$

where"$a\sim b$" means that the ratio $\frac{a}{b}$ tends to 1 as a and b tend to infinity. On the other hand, the Prime Number Theorem states:

$$(2)\ldots\pi(x)\sim\frac{x}{\log x},$$

a simpler expression than (1). The constant 1.08366 was based on Legendre's limited table for values of $\pi(x)$ up to $x=400000$. It is now known that a better estimate of (2) is:

$$(3)\ldots\ \pi(\mathrm{x})\sim\frac{x}{\log x-1}.$$

Gauss (1849) communicated in a letter to Encke that:

$$(4)\ldots\pi(\mathrm{x})\sim Li(x)$$

where $Li(x)$ is the principal value of the integral:

$$(5)\ldots Li(x)=\text{Principal value of }\int_0^x\frac{du}{\log u}$$

Indeed, interest on the density of primes has not waned for over 300 years because if (2) were proven true, then an approximate formula for the nth prime $p(n)$ would be:

$$(6)\ldots p(n)\sim n\log n.$$

In a span of 300 years, considerable effort had been expended to produce $\pi(x)$ for large values of x(Kulik, 1867); Meissel (1885); Lehmer 1959); Lagarias et al. 1985; Deleglise and Rivat ,1994; Deleglise, 1996; Gourdon and Silva 2007; Weil, 2009; Buethe and Frank, 2010). The more recent estimates of $\pi(x)$ were obtained through large – scale parallel distributed computing. We now know of the value of $\pi(10^{24})$ or 10 followed by 24 zeroes.

Hadamardand de la Valee' Poussin (1896) completely proved the asymptotic equivalence of $\pi(x)$ and $\frac{x}{\log x}$ using an earlier conjecture of Riemann (1859). Riemann averred that the non-trivial zeroes of the analytic continuation of the zeta function:

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$$(7)\ldots\zeta(s) = \int_{n=1}^{\infty} \frac{1}{n^s}.$$

occur in the complex plane of the form $\frac{1}{2} + i\,t, t \in R$. The trivial zeroes of (7) occur at the negative even integers -2, -4, -6,..., while Riemann (1859) hypothesized that the non – trivial zeroes occur in complex plane of the form $\frac{1}{2} + i\,t, t \in R$. The Riemann hypothesis remains an unsolved conjecture to date and carries a $\$1.0\ million\ prize$

Using (7), we can approximate $\pi(x)$ by the dominant term:

$$(8)\qquad R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x) = 1 + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k.k!\,\xi(k+1)},$$

Where $\mu(.)$ is the Mobius function and for small x by:

$$(9)\qquad R(x) = \frac{1}{\log x} + \frac{1}{\pi} arc\ tan\left(\frac{\pi}{\ln x}\right).$$

Rather than work through complex integration and dealing with a tough Riemann hypothesis, some scientists (Wolf (1987), Selvam (2007) and others) re-phrased the problem of estimating $\pi(x)$ in a statistical context. The self-similarity and apparent scale invariance of the density of primes $\pi(x)$ for large values of x make the estimation of the density $\frac{\pi(x)}{x}$ suited to a fractal analysis. This is the purpose of the present paper. Section 2 provides a brief overview of fractal distributions as proposed by Padua et al. (2013); Section 2 fits a fractal distribution $F_n(x,\lambda)$ to the empirical $\pi_n(x)$; Section 3 provides some numerical calculations on the fractal estimates of the density of primes; Section 4 gives the main theorem as a conclusion of the study.

2.0 Overview of Fractal Distributions and Multifractal Formalisms

The number of primes less or equal to x, $(\pi(x))$, is **not** random but unpredictable enough to merit a stochastic analysis for large x. Observations on the behavior of $\pi(x)$ for x close to a million reveal: (a) repeated patterns of irregularities at various scales, (b) self – similarity, and (c) evidence of scale invariance. For these reasons, the sequence of primes $\{x_i\}$ is treated as a random variable obeying a fractal density:

$$10\ldots\ f(x) = \frac{\lambda-1}{\theta}\left(\frac{(x)}{\theta}\right)^{-\lambda},, \qquad x \geq \theta,\ \lambda > 1,$$

Here, λ is the assumed fractal dimension of $\{\mathcal{X}\}$. The cumulative distribution function of (11) is:

$$11\ldots P(X \leq x) = 1 - \left(\frac{x}{\theta}\right)^{1-\lambda}.$$

From (10), if $(x_1),(x_2),\ldots,(x_n)$ are iidF(.), then the likelihood function is:

$$12\ldots L = \prod_{i=1}^{n} f(x_i) = \frac{(\lambda-1)^n}{\theta^n} \prod_{i=1}^{n} \left(\frac{(x_i)}{\theta}\right)^{-\lambda}$$

$$L = \left(\frac{\lambda-1}{\theta}\right)^n \cdot \left(\frac{(x_1)}{\theta}\right)^{-\lambda} \cdots \left(\frac{(x_n)}{\theta}\right)^{-\lambda}$$

or

$$13..\log L = n\ log\frac{(\lambda-1)}{\theta} - \lambda \sum_{i=1}^{n} \log(x_i)$$

The maximum – likelihood estimator of λ is then:

$$14\ldots\hat{\lambda} = 1 + n\left[\sum_{i=1}^{n} \log\left(\frac{(x_1)}{\theta}\right)\right]^{-1}.$$

For i=1, (14) reduces to:

$$15\ldots\hat{\lambda} = 1 + \frac{1}{log\left(\frac{x_i}{\theta}\right)}.$$

The maximum – likelihood estimator of λ puts equal weights on each of the values $\frac{1}{log\left(\frac{x_i}{\theta}\right)}$ but we can also use a different weighing factor as follows: Let

$$16\ldots\ F(x_\alpha) = \alpha \quad ,0 < \alpha < 1,$$

thenit follows that:

$$17\ldots 1 - \left(\frac{x_\alpha}{\theta}\right)^{1-\lambda} = \alpha,$$

and solving for λ we obtain:

$$(18)\ldots\hat{\lambda}_w = 1 - \sum_\alpha \frac{\log(1-\alpha)}{\log\left(\frac{x_\alpha}{\theta}\right)}$$

where the last term is taken over all values of α between 0 and 1. We refer to the weighted estimate $\hat{\lambda}_w(s), s = \frac{1}{\log\left(\frac{x_\alpha}{\theta}\right)}$ as the **spectrum** or **fractalspectrum** of the primes. Except at the endpoint $x = \theta$ and $x = \infty$, Equation (19) exists for all λ and is shown to be monotonic (increasing) with x (Padua, 2013).

For distributions where a single fractal dimension λ is sufficient to describe self – similarity and scale – invariance Equation is a single number and the plot $\hat{\lambda}_w(s)$ versus scale (s), where $s_\alpha = \frac{1}{\log\left(\frac{x_\alpha}{\theta}\right)}$, is a horizontal line (Padua,(2013)). We reserve discussions on a separate paper in the case when multiple rather than single fractal dimensions serve to explain data behavior more precisely.

Let p be a prime number, p = 2, 3, 5,… whose minimum value is $\theta = 2$. We first fit a monofractal probability mass function:

$$(20)\ldots f(x) = k\left(\frac{p}{\theta}\right)^{-\lambda} = \frac{k\theta^\lambda}{p^\lambda},$$
$$p = 2,3,5,\ldots,\lambda > 1$$

In order for (20) to be a proper distribution, we require that:

$$(21)\ldots\sum_p f(p) = 2^\lambda k \sum_p \frac{1}{p^\lambda} = 1$$

where the sum in (5) is taken over all the primes p. Note the close resemblance of (21) to the Riemann –zeta function and the Euler's formula for the zeta function viz:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

which is the analytic representation of the unique prime factorization theorem. More succinctly, we observe that:

$$\log\zeta(\lambda) = \sum_p \frac{1}{p^\lambda} + \sum_p \sum_2^\infty \frac{1}{mp^{m\lambda}}$$

and the first term is what appears in Equation (21).

The convergence of (21) follows from the integral test:

$$0 \le \sum_{x=2}^{\infty} f(x) = 2^\lambda k \sum_p \frac{1}{p^\lambda} \le \int_2^\infty f(x)dx$$

so that the series converges absolutely if $\int_2^\infty f(x)dx < \infty$. However,

Theorem 1: Let $0 < \delta < 1$ such that $\lambda = 1 + \delta$, then:

$$\sum_p \frac{1}{p^\lambda} < \propto$$

where the sum is taken over all primes p.

Proof: Since $\sum_p \frac{1}{p^s} \le \zeta(s) < \infty$ for $s > 1$, then result follows. ■

and:

Theorem 2. $\int_2^\infty f(x)dx < \infty \quad if\ \lambda > 1.$

Proof: $\int_2^\infty f(x)dx = (k)2^\lambda \int_2^\infty x^{-\lambda}\,dx$

$= \frac{2k}{\lambda-1} < \infty$ since $\lambda > 1$ and k $<\infty$ from Theorem 1 ■

It follows that:

$$F(x) = 2^{\lambda}k\sum_{p\le x} 1/p^{\lambda}$$

is a proper distribution iff $2^{\lambda}k = 2^{\lambda}/\sum_p \left(\frac{1}{p}\right)^{\lambda} < \infty$ which is always true since λ> 1. We approximate F(x) by

$$P(X \le x) = 1 - \left(\frac{x}{\theta}\right)^{1-\lambda}$$

since we do not know all the primes p.

Theorem 2 ensures the existence of a constant k for which F(p) is a proper distribution.

The density of primes less or equal to p is then:

(24)

$$\pi(P) = p\,F(p) = p\left[2^{1+\delta}k\sum_{q=2}^{p}\frac{1}{q^{1+\delta}}\right] = (2k)2^{\delta}\left[P\sum_{q=2}^{p^*}\frac{1}{q^{1+\delta}} + \frac{1}{p^{\delta}}\right]$$

where$^* < p$ is the prime less than p and closest to it.

Note that $\pi(P){\sim}O\left(\frac{1}{p^{\delta}}\right)$ in this formulation rather than $\pi(P){\sim}O\left(\frac{1}{\log p}\right)$ where O(.) is the "big O(h)" notation. The convergence of $\frac{\pi(P)}{\frac{1}{p^{\delta}}}$ to 1 is faster than $\frac{\pi(P)}{\frac{1}{\frac{1}{\log(p)}}}$ to 1 if $\delta \ge \frac{\log(\log p)}{\log p}, i.e.$

(26) $\frac{\log(\log p)}{\log p} \le \delta < 1.$

The quantity log(log(p)) is related to the Merten's constant M = Merten's constant = 0.2615… while the quantity log(p) is related to the Euler-Maschiaronni constant H(n)-log(n) = γ = 0.5771. Inequality (26) can be replaced by:

$\frac{M}{\gamma}$ = 0.45312 < _ < 1.

Theorem 3. The asymptotic equivalence of $\pi(p)$and $\frac{1}{p^{\delta}}$:

$$\pi(p){\sim}\frac{1}{p^{\delta}}$$

is fastest when:

$$\pi(p){\sim}\frac{1}{p^{\delta}},$$

where M = Merten's constant = 0.2615… and γ is the Euler constant $\gamma = 0.5771$ ...

This formulation enables one to search for a δ that gives closer approximation to $\frac{\pi(x)}{x}$ than 1/log(x). In the language of fractal analysis, we wish to search for the appropriate fractal dimension $= 1+\delta,\ \frac{M}{\gamma} < \delta < 1$, if there is one, or a set of fractal dimensions λ_i,if more than one fractal dimensions exist.

The statistics of the Riemann zeta zeros are a topic of interest to mathematicians because of their connection to big problems like the Riemann hypothesis, distribution of prime numbers, and others. The fractal structure of the Riemann zeta zero distribution has been studied using rescaled range analysis (Wolf, 1987).The self-similarity of the zero distribution is quite remarkable, and is characterized by a large fractal dimension of 1.9. This rather large fractal dimension is found over zeros covering at least fifteen orders of magnitude. (Wolf, 1987;Selvan, 2003)

3.0Search for a δ

We made use of the published values of the primes (MathWorld, 2012) in this paper. The maximum likelihood estimator of λ based on primes less equal to 82,989 is $\hat{\lambda}$=1.33 (see Formula (15)). The spectrum shows that larger values of λ (up to $\lambda = 1.9$) are associated with smaller scales or for larger values of x.The estimated density of primes less or equal to x is:

$$\hat{F}_n(x) = 1 - \left(\frac{x}{2}\right)^{-0.33} \qquad ,x = 1,2,\ldots,82{,}989$$

while the number of primes less or equal to x is:

$$\widehat{\Pi}_n(x) = x^{\delta}\widehat{F}_n(x)$$

whereδ is obtained by a blind – search algorithm that minimizes

$$\left|\widehat{\Pi}_n(x) - \Pi(x)\right|$$ for each x.

Table 1 shows the values of $\widehat{\Pi}_n(x)$ versus $\Pi(x)$ for small values of x (up x = 21) and for larger of x (up to x = 82, 989).

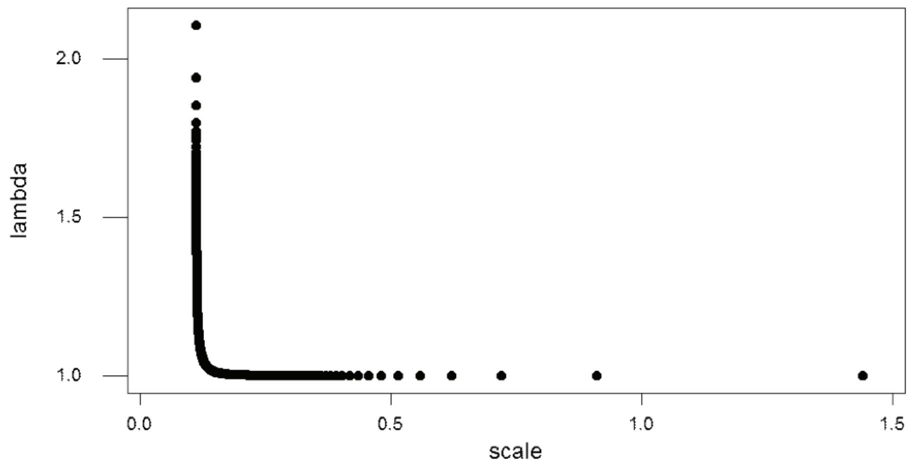

Figure 1: Fractal Spectrum of the Primes Based on primes less or equal to 82,989

Table 1: Comparison of Fractal Density Estimate and Actual Density of Primes Less or Equal to X

X	x*F(X)	Actual π(X)		X	X*F(X)	Actual π(X)
1	0	0		82970	8104	8104
2	0	1		82971	8104	8104
3	1	2		82972	8104	8104
4	1	2		82973	8104	8104
5	1	3		82974	8104	8104
6	2	3		82975	8104	8104
7	2	4		82976	8104	8104
8	3	4		82977	8104	8104
9	3	4		82978	8104	8104
10	3	4		82979	8104	8104
11	4	5		82980	8105	8104
12	4	5		82981	8105	8105
13	4	6		82982	8105	8105
14	5	6		82983	8105	8105
16	5	6		82984	8105	8105

17	5	7		82985	8105	8105
18	6	7		82986	8105	8105
19	6	8		82987	8105	8105
20	6	8		82988	8105	8105
21	7	8		82989	8105	8105

From Small to Large Scale Primes

We performed the same analysis for the number of primes less or equal to X where X = 10, $10^2,\ldots,10^{24}$, and the results are shown below using the MLE of λ earlier found ($\hat{\lambda}$=1.33).

Table 4: Fractal Estimate of the Density of Primes up to X = 10^{24}

X	ACTUAL PI(x)	F(x)	DELTA	FRACTAL ESTIMATE OF PI(x)
10.00	4	0.5323	0.9	4
100.00	25	0.7812	0.75	25
1000.00	168	0.8977	0.757	168
10000.00	1,229	0.9521	**0.77775**	1,229
100000.00	9,592	0.9776	0.79835	9,592
1000000.00	78,498	0.9895	**0.816572**	78,498
10000000.00	664,579	0.9951	**0.832097**	664,579
100000000.00	5,761,455	0.9977	**0.84519103**	5,761,455
1000000000.00	50,847,534	0.9989	0.856303944	50,847,534
10000000000.00	455,052,511	0.9995	0.8658279233	455,052,511
100000000000.00	4,118,054,813	0.9998	0.87407217668	4,118,054,813
1000000000000.00	37,607,912,018	0.9999	**0.881277237019**	37,607,912,018
10000000000000.00	346,065,536,839	0.9999	**0.887629278908300**	346,065,536,839
100000000000000.00	3,204,941,750,802	1.0000	**0.893273611345150**	3,204,941,750,802
1000000000000000.00	29,844,570,422,669	1.0000	**0.898324680333886**	29,844,570,422,669
10000000000000000.00	279,238,341,033,925	1.0000	**0.902873583024817**	279,238,341,033,925
100000000000000000.00	2,623,557,157,654,230	1.0000	**0.906993623316679**	2,623,557,157,654,050
1000000000000000000.00	24,739,954,287,740,800	1.0000	**0.910744410637815**	24,739,954,287,739,900
10000000000000000000.00	234,057,667,276,344,000	1.0000	**0.914174900293341**	234,057,667,276,344,000
100000000000000000000.00	2,220,819,602,560,910,000	1.0000	**0.917325669563945**	2,220,819,602,560,870,000
1000000000000000000000.00	21,127,269,486,018,700,000	1.0000	**0.920230639188459**	21,127,269,486,018,000,000
10000000000000000000000.00	201,467,286,689,315,000,000	1.0000	**0.922918389153279**	201,467,286,689,306,000,000
100000000000000000000000.00	1,925,320,391,606,800,000,000	1.0000	**0.925413174858688**	1,925,320,391,606,800,000,000
1000000000000000000000000.00	18,435,599,767,349,200,000,000	1.0000	**0.927735719847600**	18,435,599,767,348,800,000,000

Except for the last estimate of the density of primes less or equal to 10^{24}, the fractal estimates of the density of primes less or equal to X are identical with the actual count of the primes less than or equal to X. From X = 10^{14} to 10^{24}, the fractal estimate of the number of primes less or equal to x is equal to:

$$\widehat{\pi(X)} = X^{\delta} \text{ as } X \to \infty \qquad \text{since} F(x) \to 1.$$

Figure 2 shows the fractal spectrum of the exponents used:

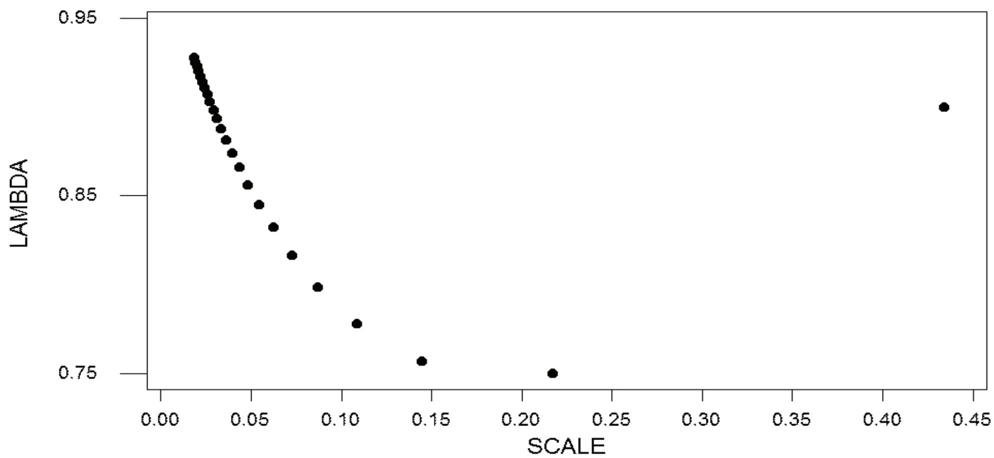

Figure 2: Fractal Spectrum: Lambda versus scale (scale = 1/ln(X)).

We note that the fractal spectrum confirms earlier conjectures that the fractal dimension is quite large ($\lambda = 1.9$ or higher) for larger value of x, i.e. $\delta = \lambda - 1 = 0.9$ or higher.

Table 3: Percentage Errors for Estimating the Primes Less or Equal to X using the Fractal Estimate and the PNT

X	Fractal Relative Error	PNT Relative Error
10	5.70%	8.57%
10^2	1.18%	13.14%
10^3	0.27%	13.83%
10^4	0.03%	11.66%
10^5	0.00%	9.45%
10^6	0.00%	7.79%
10^7	0.00%	6.64%
10^8	0.00%	5.78%
10^9	0.00%	5.10%
10^{10}	0.00%	4.56%
10^{11}	0.00%	4.13%
10^{12}	0.00%	3.77%
10^{13}	0.00%	3.47%
10^{14}	0.00%	3.21%
10^{15}	0.00%	2.99%
10^{16}	0.00%	2.79%

10^{16}	0.00%	2.79%
10^{17}	0.00%	2.63%
10^{18}	0.00%	2.48%
10^{19}	0.00%	2.34%
10^{20}	0.00%	2.22%
10^{21}	0.00%	2.11%
10^{22}	0.00%	2.02%
10^{23}	0.00%	1.93%
10^{24}	0.00%	1.84%
Average	0.30%	5.18%

It is interesting to note that the PNT estimate for the number of primes less or equal to 10^{25} is: 173,717,792,761,301,000,000,000. The ratio of the fractal estimate and the PNT estimate is 99.86900% at this X.

4.0 Conclusion

The numerical calculations above serve to demonstrate three(3) key ideas which we now state as Theorems:

Theorem 4. Given ε > 0, there existsa _ > 0 such that

$$|\pi_\lambda(N) - \pi(N)| < \varepsilon \quad \text{as } N \to \infty \text{ and } \lambda = 1 + \_.$$

Proof: Let:

$$\pi_\lambda(N) = N^{-}(1 - N^{1-\lambda})$$

whereλ = fractal dimension of π(N)> 1. Now,

$$|\pi_\lambda(N) - \pi(N)| = \left|\pi_\lambda(N) - \frac{N}{logN} + \frac{N}{logN} - \pi(N)\right|$$

$$\leq \left|\pi_\lambda(N) - \frac{N}{logN}\right| + \left|\frac{N}{logN} - \pi(N)\right|$$

$$\leq \left|\pi_\lambda(N) - \frac{N}{logN}\right| + \varepsilon/2$$

by Hadamard-de la ValeePoussin

But,

$$\left|\pi_\lambda(N) - \frac{N}{logN}\right| = \left|N^{\square}(1 - N^{1-\lambda}) - \frac{N}{logN}\right|$$

$$\leq \left|N^{\square} - \frac{N}{logN}\right| \text{ because } N^{1-\lambda} \to 0 \text{ as } N \to \infty$$

$$\leq \varepsilon/2 \text{ with } \square \geq 1 - \frac{\log(\log(N))}{\log N}$$ ■

Theorem 5. Let :

$$\hat{\lambda} = 1 + n(\sum \log(x))^{-1}$$

be the maximum-likelihood estimator of λ for the fractal distribution:

$$f(x) = \frac{\lambda-1}{\theta}\left(\frac{(x)}{\theta}\right)^{-\lambda} \quad x \geq \theta, \lambda > 1$$

,then, $\hat{\lambda}$converges to λ in probability as n→∞ .

Proof: (see Graybill, "Linear Models", pp. 34-37).■

As a corollary, we state our main conclusion that:

Corollary:$\pi(x) \sim x^{\delta}$ where _ > 1 – $\frac{\log(\log(x))}{logx}$ as x→∞

Finally, we conclude that x/log(x) is significantly different fromπ(x) for small values of x but there is a fractal dimension λ for which π(x) will be close to $\pi_\lambda(x)$ even for small values of x. It follows that the fractal approximation of the density of primes less or equal to x is uniformly superior to the classical Prime Number Theorem approximation.

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