Construction of a Simple Multifractal Spectrum as an Alternative to the Legendre Spectra in Multifractal Formalisms

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Abstract

It is important to realize at the beginning of a statistical analysis whether the data are from a monofractal or multifractal distribution because the methods of analysis are different for each. In seismic sequence analysis, for instance, the monofractal method uses the R/S and DFA (range-scale and detrended fluctuation analysis, respectively) techniques while the multifractal formalism uses the partition function technique (PFT) and the Legendre spectra outputting three parameters: maximum α_0 of the spectrum, asymmetry B and width W of the curve (Lapenna et al. (2003)). In this paper, we introduce a simple test of mono or multifractality of data sets. The test is based on fitting a power-law distribution to a random sample obtained from some unknown distribution G(.). For each quantile, a fractal dimension $\hat{\lambda}_{\alpha_k}$ is obtained. This corresponds to the Legendre spectra or multifractal spectra. A regression function is fitted to the points (tk, $\hat{\lambda}_{\alpha_k}$) and the slope b of this line is tested. If b = 0, then the observations are deduced to have come from a monofractal distribution f(x). The paper proposed a test for monofractality which, in effect, also tests for multifractality or non-fractality of a set of observations. For monofractal observations, the proposed new multifractal spectral analysis revealed a single point (singularity at a point) while for multifractal observations, a single-humped continuous quadratic function is observed. The parameters of the quadratic function are interpreted as the measure of asymmetry (B), ruggedness (C) and width (W). The new proposed multifractal spectrum function is easier to calculate and is consistent with the more complicated Legendre spectrum proposed in the literature.

Keywords: monofractal, multifractal statistical observations, power-law

1.0 Introduction

Telesca, Lapenna and Macchiato (2003) analyzed the earthquake sequences from 1986 to 2001 in three different seismic zones in Italy by means of monofractal and multifractal methods and demonstrated that the multifractal analysis has given superior quantitative information about the "complexity" of the seismic series. The use of fractal and multifractal statistical tools in seismic analysis was pioneered by Smalley et al. (1987); Kagan (1994) who reviewed experimental evidences for earthquake scale invariance. In earlier works, a single Hurst exponent (fractal dimension λ) was used in conjunction with modeling seismic patterns using the Fano factor and the Allan factor (Telesca, Lapenna and Macchiato(2001)). While one scaling exponent can completely describe a monofractal, not all irregular data are homogeneous in the sense that they have the same scaling properties. The need for more than one scaling exponent to

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describe the scaling process uniquely may require more than one exponent (hence, multifractal). Multifractals are intrinsically more complex and inhomogeneous than monofractals; they can be decomposed into many – possibly infinite – subsets characterized by different scaling exponents (Stanley et al. 1999).

Mathematically, a monofractal distribution is described by the power – law probability density:

$$f(x;\lambda) = \frac{\lambda-1}{\theta}\left(\frac{x}{\theta}\right)^{-\lambda}, \lambda > 1, x \geq \theta$$

with fractal dimension $(\lambda - 1)$. By abuse of notation, we shall refer to λ as the fractal dimension rather than $(\lambda - 1)$. The fractal dimension λ describes the "space-filling" property of the observations relative to a scale. Thus, in a mono fractal system, the self – similarity of the objects is repeated at various scales with the same space – filling property (Padua, 2013).

On the other hand, multifractal distributions are a mixture of monofractal distributions with dimensions $\lambda_1, \lambda_2, \ldots, \lambda_k$. Following Tukey's model (1972), we write:

$$(2)\ldots f(x) = \alpha_1 f_1(x,\lambda_1) + \alpha_2 f_2(x,\lambda_2) + \cdots + \alpha_k f_k(x,\lambda_k)$$

with $\sum_{i=1}^{k} \alpha_i = 1$, $\alpha_i > 0$. Theoretically, $k \to \infty$ and $\alpha_1, \alpha_2, \ldots, \alpha_k, ..$ is a sequence of real numbers between 0 or 1 such that $\sum_{i=1}^{\infty} \alpha_i = 1$.

Given a random sample $x_1, x_2, \ldots, x_n$ from f(x), we are asked to determine whether they came from (1) or from (2). Since the data are highly irregular and heterogeneous, the usual Fisher's linear discriminant analysis (FLD) or quadratic discriminant analysis will not work here (Johnson and Wichern, 2000).

It is important to realize at the beginning of a statistical analysis whether the data are from a monofractal or multifractal distribution because the methods of analysis are different for each. In seismic sequence analysis, for instance, the monofractal method uses the R/S and DFA (range-scale and detrended fluctuation analysis, respectively) techniques while the multifractal formalism uses the partition function technique (PFT) and the Legendre spectra outputting three parameters: maximum α_0 of the spectrum, asymmetry B and width W of the curve (Lapenna et al. (2003)).

In this paper, we introduce a simple test of mono or multifractality of data sets.

2.0 Fractal Statistical Formalisms

A random sample $x_1, x_2, \ldots, x_n$ is said to obey a fractal distribution if the probability density function describing their distribution obeys:

$$(3) \quad f(x;\lambda) = \frac{\lambda-1}{\theta}\left(\frac{x}{\theta}\right)^{-\lambda}, \lambda > 1, x \geq \theta$$

It is shown in Padua (2013) that the maximum likelihood estimators of λ and θ are respectively.

$$(4) \quad \hat{\lambda} = 1 + n\left(\sum_{i=1}^{n} \log\left(\frac{x_i}{\hat{\theta}}\right)\right)^{-1}$$

$$(5) \quad \hat{\theta} = min\{x_1, x_2, \ldots, x_n\}.$$

The maximum-likelihood estimator of λ is unbiased for λ and is a function of a complete, sufficient statistic. It follows from the Lehmann and Scheffe's theorem that it is also the unique minimum variance unbiased estimator of λ (UMVUE).

We note that both $\mu = E(x)$ and $\sigma^2 = var(x)$ may not exist for fractal observations so that $\bar{x}$ and s^2 are meaningless estimators of μ and σ^2, respectively. Consequently, the law of large numbers $(\bar{x} \to \mu)$ and the central limit theorem do not work. We have shown in an earlier paper that the median, $\tilde{\mu}$, and median absolute deviation, $\tilde{\sigma}$, can be considered in place of μ and σ which do not exist.

A practical approach suggested in the literature to estimate λ is to plot $\log f(x)$ versus $log\left(\frac{x}{\theta}\right)$ and to use the slope of the line as estimators of λ. This could be heuristically argued by taking the logarithm of both sides of (3):

$$(6)\quad \log f(x) = log\left(\frac{\lambda-1}{\theta}\right) - \lambda\, log\left(\frac{x}{\theta}\right).$$

However, the constant, $log\left(\frac{\lambda-1}{\theta}\right)$, also contains the unknown parameters λ and θ (Palmer,1992). Furthermore, a careless use of regression method to estimate λ may lead to erroneous conclusions. For instance, if x were normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$,

$$(7)\quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, -\infty < x < \infty$$

then:

$$(8)\quad log f(x) = log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}x^2.$$

Equation (8) says $\hat{\lambda} = \frac{1}{2}$ but x is not even distributed according to a fractal distribution.

3.0 The Proposed Fractality Test

Let $x_1, x_2, \ldots, x_n$ be iid G(.) where G(.) is an unknown absolutely continuous function with respect to a Lebesque measure. Without loss of generality, we assume that $x_i > 0$ and in fact, $x_i > \theta$ for i= 1, 2, …., n. The idea is to fit a fractal distribution $f(x)$ to the quantile of the distribution G(.).

Data sets may exhibit monofractal, multifractal or non – fractal behavior. Observations $\{x_i\}$ that exhibit monofractality obeys the fractal distribution (3) with a single defining fractal dimension λ; multifractal observations obey (3) but different λ_i's work for different scales viz. small scale to larger scales. Finally, G(.) may not even be a fractal distribution in which case, different λ_i work for x_i so that even for the same scale, different λ_i's will be found.

Indicators of Monofractality

Let $x_{(\alpha)}$ be the α^{th} quantile of G(.):

$$(9)\ldots G\left(x_{(\alpha)}\right) = \alpha$$

At each of α^{th} quantile of G(.), we fit a power law distribution F(.) such that:

$$(10)\ldots G\left(x_{(\alpha)}\right) = F\left(x_{(\alpha)}\right) = \alpha,$$

or equivalently, obtain:

$$(11)\ldots \lambda_{(\alpha)} = 1 - \frac{\log(1-\alpha)}{log\left(\frac{x_{(\alpha)}}{\theta}\right)}, \text{ for all } \alpha \in (0,1).$$

Denote the empirical quantiles by $X_{(\alpha_k)}$ where $\alpha_k = \frac{k}{n}$, $1 \le k \le n-1$. An estimate of λ can be obtained from (11):

$$(12)\ldots \hat{\lambda} = \frac{1}{n-1}\sum_{k=1}^{n-1} \lambda_{(\alpha_k)}.$$

With $\theta = min\{x_1, \ldots, x_n\}$, we know that $\log(1-\alpha) \le log\frac{x_\alpha}{\theta}$ so that (11) is a monotone function of α (in fact, monotonic non-decreasing). We state this as a result:

Result 1: Let $\lambda_{(\alpha)} = 1 - -\frac{\log(1-\alpha)}{log\left(\frac{x_{(\alpha)}}{\theta}\right)}\ for\ all\ \alpha \in (0,1)$, then $\lambda_{(\alpha)}$ is a monotonic non – decreasing function of x.

Proof:

Let $x_1 \le x_2$ then $F(x_1) \le F(x_2)$ so $\alpha_1 \le \alpha_2$. By the monotone property of the logarithm, $log(1-\alpha_1) \geq log(1-\alpha_2)$. Obviously, $log\left(\frac{x_1}{\theta}\right) \leq log\left(\frac{x_2}{\theta}\right)$. It follows that:

$$\frac{log(1-\alpha_1)}{log\left(\frac{x_1}{\theta}\right)} \geq \frac{log(1-\alpha_2)}{log\left(\frac{x_1}{\theta}\right)}$$

Or

$$\frac{log(1-\alpha_2)}{log\left(\frac{x_1}{\theta}\right)} \geq \frac{log(1-\alpha_1)}{log\left(\frac{x_1}{\theta}\right)}.$$

Since $log\left(\frac{x_2}{\theta}\right) \geq log\left(\frac{x_1}{\theta}\right)$, we obtain:

$$\frac{log(1-\alpha_2)}{log\left(\frac{x_2}{\theta}\right)} \leq \frac{log(1-\alpha_2)}{log\left(\frac{x_1}{\theta}\right)} \leq \frac{log(1-\alpha_1)}{log\left(\frac{x_1}{\theta}\right)}$$

Thus:

$$\frac{log(1-\alpha_2)}{log\left(\frac{x_2}{\theta}\right)} \leq \frac{log(1-\alpha_1)}{log\left(\frac{x_1}{\theta}\right)}$$

$$1-\frac{log(1-\alpha_2)}{log\left(\frac{x_2}{\theta}\right)} \geq 1-\frac{log(1-\alpha_1)}{log\left(\frac{x_1}{\theta}\right)}$$ as desired ■

It is clear that λ(x) is a one-to-one function. In effect, we have succeeded in establishing a one-to-one , monotonic increasing transformation of the random variables X = { $x_1, x_2, \ldots, x_n$} to the space Ω of the fractal dimensions λ greater than or equal to 1. The advantage of working with the space Ω is that the fluctuations in the values of x can be more clearly seen in Ω than in the original set of observations. The transformation can be viewed as a powerful microscope that magnifies small differences in the original values.

Let $s_k = \frac{1}{\ln\left(\frac{x_{\alpha_k}}{\theta}\right)}, \quad k = 1, 2, \ldots, n-1.$

The series $\hat{\lambda}_{(\alpha_k)}$ is a discrete time series indexed by k (or equivalently, s_k). Define s_k to be the scale function.

Result 2. Let $\hat{\lambda}_{\alpha_k} = 1 - \frac{\log(1-\alpha_k)}{log\left(\frac{x_{\alpha_k}}{\theta}\right)}$, k =1, 2, …, n-1 be the estimates of λ obtained by fitting the a^{th} quartiles of a distribution G(.) to a power – law distribution F(.). If the time series $\{\hat{\lambda}_{\alpha_k}\}$ is a trendless series for k = 1,2,…,n, then $\hat{\bar{\lambda}} = \bar{\lambda}$, and the original observations come from a monofractal distribution with fractal dimension equal to $\bar{\lambda}$. Equivalently, let:

$\hat{\lambda}_{\alpha_k} = a + b\, s_k + \varepsilon_k \quad k = 1, 2, \ldots, n-1$ be a model for the fractal dimensions as a function of the scale s_k. If b = 0, then $\hat{\lambda} = \hat{a}$ is an estimate of λ and the data come from a monofractal distribution F(.).

Proof:

From:

$$b = \frac{\sum_{k=1}^{n-1}(t_k - \bar{t})(\lambda_{\alpha_k} - \bar{\lambda})}{\sum_{k=1}^{n-1}(t_k - \bar{t})^2}$$

We conclude that b = 0 if $\{\lambda_{\alpha_k}\}$ is monotonic non – decreasing since:

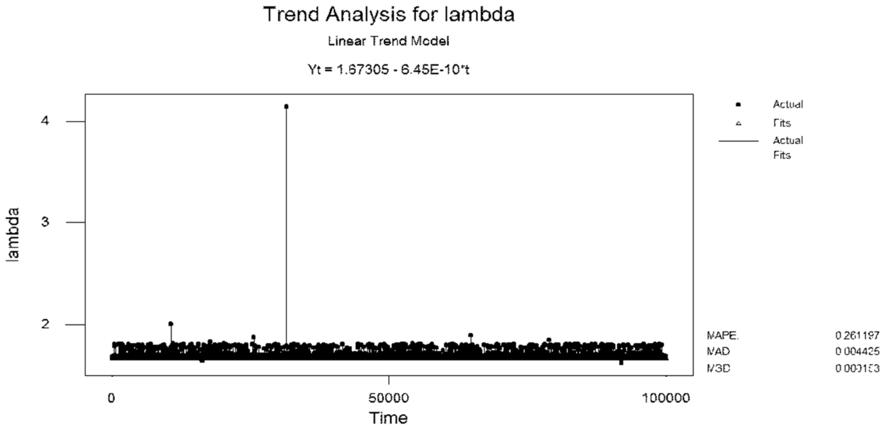
$$\sum_{k=1}^{n-1}(t_k - \bar{t})(\lambda_{\alpha_k} - \bar{\lambda}) = 0.$$

From:

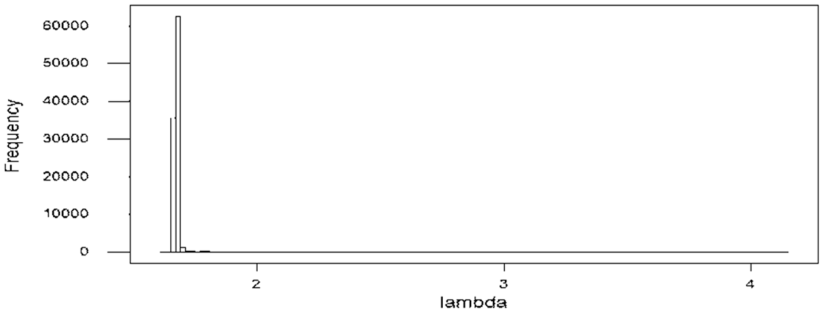
$$a = \bar{\lambda} - b\bar{t}$$

We conclude that $a = \bar{\lambda} = \hat{\lambda}$. From these, we deduce that there is one λ that works for all scales s_k , hence, the distribution is monofractal. ■

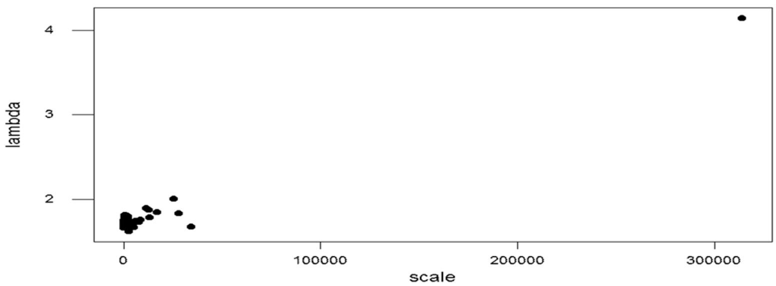
For monofractal distributions, the distribution of the fractal dimensions λ_k is almost a singular distribution with $P(\lambda = \lambda_0) = 1$. We generated 100,000 monofractal random variables with λ = 1.67. The trend analysis is shown in the figure below:



The histogram for the values of the fractal dimensions is shown below:



while the graph of the fractal dimensions versus scale is shown below:



Indicators of Multifractality

The usual way to deal with multifractal observations is to use the multifractal formalism. Many authors have developed the concept of multifractality and its formalism applying it to several fields of scientific research (Davis et al.(1994); Diego et al.(1999); Peng C-K et al. (1995)).

A partition function $Z(q,\varepsilon)$ is defined which essentially sums up the probability measures $\mu_j(\varepsilon)$ depending on ε or scale of the boxes used to cover the sample. This partition function obeys a power law :

$$Z(q,\varepsilon) = k\varepsilon^{T(q)}.$$

For a certain box j, the probability measure is assumed to obey:

$$\mu_j(\varepsilon) = k_1\varepsilon^{\alpha j}.$$

The cardinality of all subsets S_α having the same value of α is denoted by $N_\alpha(\varepsilon)$ and obeys:

$$N_\alpha(\varepsilon) = k_2\varepsilon^{-f(\alpha)}.$$

The curve f(α) is called the **Legendre spectrum** and is a single-humped function for a multifractal and reduces to a point for a monofractal. In order to make quantitative statements about possible differences in the Legendre or multifractal spectra coming from different signals, it is usual to fit a quadratic function:

$$(f(\alpha)) = A(\alpha-\alpha_0)^2 + B(\alpha- \alpha_0) + C$$

where α_0 is the position of the x-coordinate of the maxima. If α_0 is low, the signal is correlated and the underlying process loses fine structure, becoming more regular in appearance. Thus, we shall refer to this as a **ruggedness measure**. The parameter B is a measure of **asymmetry** of the curve: zero for symmetric shapes, positive or negative for left-skewed or right-skewed shapes respectively. The asymmetry parameter B gives information on the dominance of low or high fractal exponents with respect to the other. A right-skewed spectrum denotes relatively strongly weighted high fractal exponents, corresponding to fine structures, and low ones (more smooth looking) for left skewed spectra.

The **width** of the spectrum is defined as:

$$W = \alpha_{max} - \alpha_{min}$$

where $f(\alpha_{max}) = f(\alpha_{min}) = 0$.

The width measures the length of the fractal exponents in the signal: the wider this range, the richer the signal is in structure.

Multifractal Spectra via λ(s)

Our method proposes to replace f(.) in the multifractal formalism by the function:

$$\lambda(s) = 1 - \log(1- \alpha\ )s$$

where the scale function $s = \frac{1}{\log(\frac{x}{\theta})}$. We have demonstrated that λ(s) is 1-1, a monotone increasing function of x, and therefore qualifies as a spectrum of the original signals x. If λ(s) reduces to a point or a cluster of points, then we conclude that we have a monofractal, otherwise, if it shows a single-humped function, then we conclude the presence of a multifractal set of observations. In a similar vein, we can then express λ(s) as:

$$\lambda(s) = A(s-s_0)^2 + B(s- s_0) + C$$

parameters A,B and C = $\lambda(s_0)$ and their usual interpretations. Note the simplicity of the approach as opposed to the classical multifractal formalisms.

Test Algorithm

(1) Sort $x_1, x_2, \dots, x_n$ from smallest to highest. Denote the sorted values by $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$.

(2) Assign the weights $\alpha_i = i = 1, 2, \dots, n$. Remove the last observations. (highest observation). Let $\theta = x_{(1)}$.

(3) Compute $\hat{\lambda}$ based on (4) and (5).

(4) Do a time series plot for $\hat{\lambda}_{\alpha_i}$, $i = 1, 2, \dots, n-1$, and regress $\hat{\lambda}_{\alpha_*}$ versus t where t is defined in Result 2, $\hat{\lambda}_{\alpha_*} = a + b\ t$. If Ho: b = 0, then conclude $\{x_1, \dots, x_n\}$ came from a power – law distribution with fractal dimension λ; else, compute the multifractal spectra. If there is a single-humped continuous function, then the observations come from a multifractal distribution with spectral parameters A,B and C earlier mentioned.

4.0 Simulation Results

We performed three simulation experiments using a monofractal distribution with λ = 1.63 and n = 1000; a multifractal distribution with $\lambda_1 = 1.33$, $\lambda_2 = 1.67$with $n_1 = n_2 = 1000$; and a non-fractal distribution , namely, the standard normal distribution N(0,1).

Result 1: Figure 1 shows the data displayed as a time series of values from a monofractal distribution. Spikes are noted in the graph of the time series

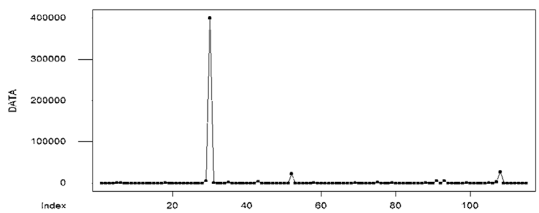


Figure 1: Time Series of Observations from a monofractal

Table 1 summarizes the descriptive statistics for the estimated value of the fractal dimension.

Table 1: Descriptive Statistics for the Estimated Fractal Dimension

Variable	N	Mean	Median	StDev
lambda	999	1.6227	1.6357	0.0840

Figure 2 shows the scatterplot of the estimated values of the fractal dimension versus the reciprocal of the logarithm of the data divided by the minimum value.

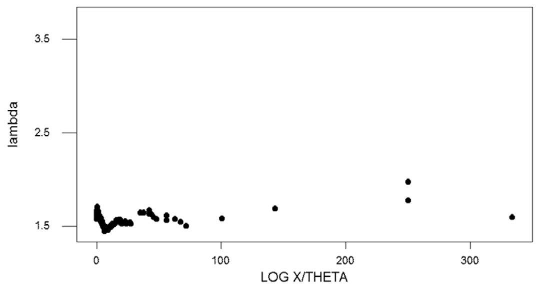


Figure 2: Scatterplot of lambda versus 1/log(x/theta)

The scatterplot shows no trend in the values of the fractal dimension as a function of the scale. This is supported by the regression analysis performed on lambda versus the scale as reflected in Table 2.

Table 2: Regression Analysis of Lambda versus Scale

The regression equation is

lambda = 1.62 -0.000000 LOG X/THETA

Predictor	Coef	SE Coef	T	P
Constant	1.62059	0.00166	975.54	0.000
LOG X/TH	-0.00000008	0.00009366	-0.00	0.999

Since b = 0, we conclude that the data came from a monofractal distribution with fractal dimension approximately 1.6227 (as opposed to the theoretical value of 1.63).

The simulation was repeated 100 times for n = 1000 observations from the monofractal distribution with essentially the same results. The statistics for the 100 replications are displayed in Table 3:

Table 3: Descriptive Statistics for 100 replications on a monofractal

Statistic	Mean	SE(Mean)
Lambda	1.6271	0.06420
Regression Coeff.	0.0001	0.00003

Result 2: Figure 3 shows the plot of the data obtained from a normal distribution with mean 0 and variance 1 (N(0,1)). The absolute values of the observations were obtained in order to put the context in fractal statistics. We note the massive "spikeness" of the data as opposed to the monofractal observations obtained earlier.

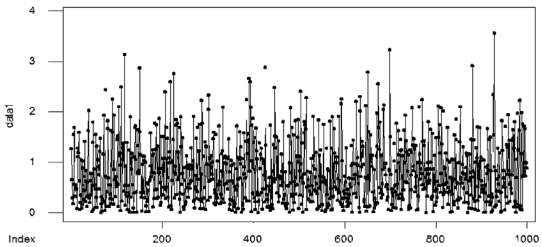


Figure 4: Time series of the estimated values of lambda

The spikes are likewise inherited by the estimated values of lambda showing that no single fractal dimension can be used to fully describe the data fluctuations. In fact, a different exponent can be found for each observation. This is illustrated in Figure 4.

Table 3 shows the descriptive statistics obtained for the estimated values of the fractal dimensions when 10,000 standard normal variates were generated and the corresponding fractal dimensions computed each for 30 different sampling runs.

Table 3: Descriptive Statistics for the fractal dimension

Variable	N	Mean	Median	TrMean	StDev	SE Mean
lambda1	10000	1.0857	1.0623	1.0772	0.0802	0.0008
lambda2	10000	1.1270	1.0942	1.1151	0.1149	0.0011
lambda3	10000	1.0974	1.0713	1.0879	0.0903	0.0009
lambda4	10000	1.1124	1.0827	1.1016	0.1030	0.0010
lambda5	10000	1.1188	1.0879	1.1075	0.1083	0.0011
lambda6	10000	1.1370	1.1023	1.1244	0.1226	0.0012
lambda7	10000	1.0858	1.0623	1.0773	0.0803	0.0008
lambda 8	10000	1.1015	1.0744	1.0916	0.0938	0.0009
lambda9	10000	1.1186	1.0877	1.1074	0.1081	0.0011
lambda10	10000	1.0948	1.0693	1.0855	0.0879	0.0009
lambda11	10000	1.0859	1.0625	1.0774	0.0802	0.0008
lambda12	10000	1.1118	1.0824	1.1011	0.1024	0.0010
lambda13	10000	1.1366	1.1017	1.1239	0.1228	0.0012
lambda14	10000	1.0913	1.0665	1.0823	0.0850	0.0009
lambda15	10000	1.1118	1.0823	1.1011	0.1023	0.0010
lambda16	10000	1.1045	1.0767	1.0944	0.0963	0.0010
lambda17	10000	1.1143	1.0842	1.1034	0.1045	0.0010
lambda18	10000	1.1030	1.0756	1.0930	0.0951	0.0010
lambda19	10000	1.0850	1.0618	1.0765	0.0795	0.0008
lambda20	10000	1.1091	1.0803	1.0986	0.1002	0.0010

(Table 3 continued)

Variable	N	Mean	Median	TrMean	Std Dev	SE(mean)
lambda21	10000	1.0849	1.0616	1.0764	0.0796	0.0008
lambda22	10000	1.1069	1.0784	1.0966	0.0985	0.0010
lambda23	10000	1.0850	1.0618	1.0765	0.0795	0.0008
lambda24	10000	1.0850	1.0617	1.0765	0.0796	0.0008
lambda25	10000	1.0851	1.0618	1.0766	0.0796	0.0008
lambda26	10000	1.1109	1.0817	1.1002	0.1016	0.0010
lambda27	10000	1.1169	1.0864	1.1058	0.1067	0.0011
lambda28	10000	1.1151	1.0851	1.1041	0.1050	0.0010
lambda29	10000	1.1204	1.0892	1.1090	0.1094	0.0011
lambda30	10000	1.1070	1.0786	1.0967	0.0984	0.0010
Variable	N	Mean	Median	TrMean	StDev	SE Mean
lambda	30	1.1050	1.1069	1.1041	0.0157	0.0029

A plot of the estimated values of lambda versus the reciprocal of log (y/θ) or the scale, shows an almost vertical line (of almost infinite slope). Figure 5 shows the situation.

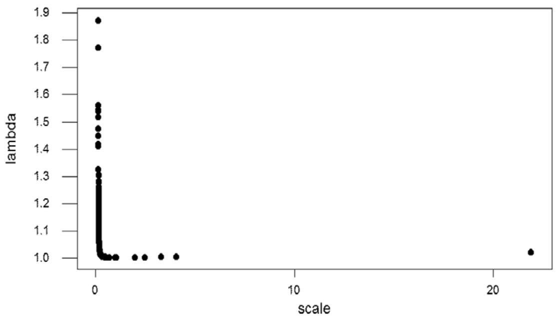


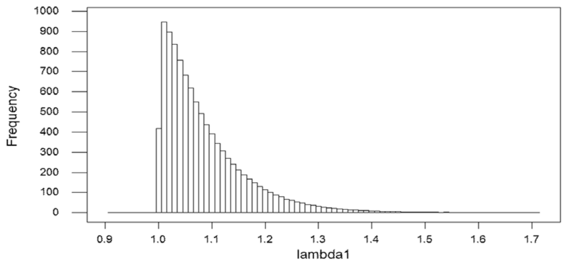
Figure 5: Plot of lambda versus scale for normal variates.

The histogram of the estimated values of lambda for 10,000 samples from the standard normal distribution shows a definite pattern of exponentially distributed random variates as shown in Figure 6 and whose density function can be written as:

(13) $g(\lambda) = Ae\text{-}k\lambda = ke\text{-}k(\lambda\text{-}1)\ \lambda > 1$

where A = kek. For the standard normal random variates whose mean fractional dimension is 1.1050, the value of k is k = 1/0.1050 = 9.5238. It follows that (13) becomes:

(14) $g(\lambda) = 9.5238\ e\text{-}9.5238(\lambda\ \text{-}1),\ \lambda > 1$



Result 3: Two fractal distributions were mixed. One of the fractal distributions has a dimension equal to 1.33 while the other has a fractal dimension of 1.67. One thousand observations (1000) were each taken from the two distributions. Figure 7 shows the time series plot of the mixed observations. Note that the time series plot shows a signature plot for fractal or multifractal observations.

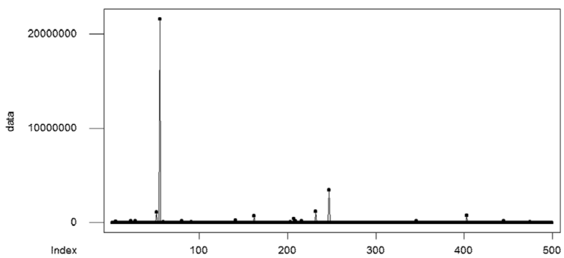


Figure 7: Time Series Plot of the mixed fractals

Figure 8 shows the histogram of the computed values of lambda. Notice that the probability distribution of the random variables representing the fractal dimensions , lambda, appear to obey a power law distribution.

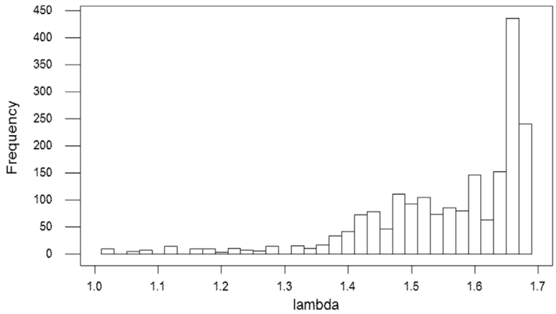


Figure 8: Histogram of the values of lambda

Figure 9 shows a scatterplot of lambda versus scale. Note that the function appears to be a single humped continuous function up to scales between 3 to 4. This illustrates that the data may have come from a mixture of two fractal distributions.

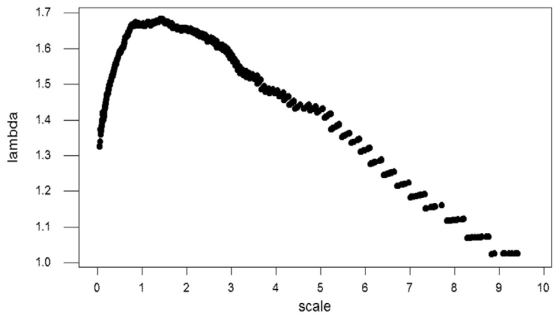


Figure 9: Plot of the multifractal spectra

Lapenna et al. (2004), using a different method for constructing the multifractal spectra, averred that a single-humped shape is typical of all multifractal signals.

Quadratic Fit

Using the usual least-squares method, we fitted a quadratic function to the multifractal spectrum with λ as the response variable and s, s^2 as the predictors. The resulting quadratic function was differentiated to obtain s_0. The quadratic fit around $s_0 = 2.25$ is shown below:

The regression equation is
lambda = 1.62 + 0.00009 (scale-s0) - 0.0173 (scale-s0)^2

1999 cases used 1 cases contain missing values

Predictor	Coef	SE Coef	T	P
Constant	1.62116	0.00242	671.13	0.000
scale-s0	0.000088	0.001162	0.08	0.939
(scale-s	-0.0173221	0.0003471	-49.90	0.000

S = 0.07067 R-Sq = 69.9% R-Sq(adj) = 69.9%

Here A = -0.0173, B = 0.00009, C = 1.62, showing slight left-skewness (B =.00009) of the signals but with defined fine structures (s_o = 2.25). The width is W = 9.4221-0.0365 = 9.3856 which is quite wide showing that the signals are quite rich in structure. The Hurst exponents (λ) themselves show that the series behaviour is characterized by persistent behaviour: low(high) values are more likely to be followed by low (high) values of the signals.

Further quadratic fit on the unusual observations found after the first regression pass showed that the fractal dimension λ = 1.33 had B = -0.00958 demonstrating that these set of observations were in fact right skewed but those belonging to the fractal dimension λ = 1.67 had positive B or left skewed appearance. On the average, the overall effect is to have slightly left skewed signals with wide spectrum.

5.0 Conclusion

The paper proposed a test for monofractality which, in effect, also tests for multifractality or non-fractality of a set of observations. For monofractal observations, the proposed new multifractal spectral analysis revealed a single point (singularity at a point) while for multifractal observations, a single-humped continuous quadratic function is observed. The parameters of the quadratic function are interpreted as the measure of asymmetry (B), ruggedness (C) and width (W). The new proposed multifractal spectrum function is easier to calculate and is consistent with the more complicated Legendre spectrum proposed in the literature.

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