Some Results on Multifractal Spectral Analysis

*Roberto N. Padua, [1]Efren O. Barabat, [2]Mark S. Borres and [1]Randy K. Salazar

Abstract

A multifractal spectrum, based on an earlier paper and different from the Legendre multifractal spectrum (Padua et al. 2013) was examined in this paper. The examination yielded interesting results which enhanced the utility of the developed $\lambda(s)$-multifractal spectrum in analyzing real data. One of the results show that a mixture of several monofractal observations can be represented as a single monofractal distribution but whose spectrum is different from the spectrum of the original data. Thus, high fractal dimensional distributions can be infinitely decomposed into component monofractal dimensions. Further, we also show that given a multifractal set of observations, observations that fall on smaller scales obey a normal distribution. The study ends by providing possible avenues for future research particularly in the area of analytic number theory in relation to the Riemann hypothesis about the distribution of primes.

Keywords: multifractal, monofractal, $\lambda(s)$- spectrum, Legendre spectrum

1.0 Introduction

The utility of multifractal analysis in the analysis of seismic data in Italy was demonstrated by seismic data in Italy was demonstrated by Lapenna et al. (2003), in the Philippines by Panduyos and Padua (2013), and in other countries by various authors. Of these multifractal models of seismic data, the main tool used was Legendre's multifractal spectrum which essentially involves finding a sequence of multifractal manifolds which can be expressed in terms of power laws. In Padua and Barabat (2013) a simpler multifractal spectrum $\lambda(s)$ was found useful in fractal data analysis.

Multifractal probability distributions are define as mixture of m monofractal distributions in the sense of Tukey (1972):

(1) $f(x) = w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x), x \geq \theta$

where:

(2) $f_i(x) = \left(\frac{\lambda_i - 1}{\theta}\right)\left(\frac{x}{\theta}\right)^{-\lambda_i} \quad, i = 1,2,3,4,\ldots,m \quad \sum_{i=1}^{m} w_i = 1 \quad, \quad w_i \geq 0.$

The exponents λ_i are the fractal dimensions which determine the information- filling property of an observation x_i from this distribution. In several papers, authors claimed that fractal observations are, in fact, more pervasive in real – life than normal observations (Selvam (2008)), Lapenna et al (2003), Padua et al (2003), and Salazar (2013). As such, fractal distributions need to be examined more closely and classical normal-based methods, reviewed.

A useful device for examining multifractal observations is the multifractal spectrum. The current multifractal spectrum in use is the Legendre' spectrum but its application is largely confined to scientists in specialized fields because of its complexity. Padua (2013) suggested a simpler version of a multifractal spectrum, namely:

(3) $\lambda(s) = 1 - \frac{\log(1-\alpha)}{\log\left(\frac{x}{\theta}\right)} \quad, \theta \geq x, \quad \alpha = F(x)$

*Fractal Statistics Expert
[1]College of Engineering
[2]Research Department
University of San Jose – Recoletos

$$\lambda(s) = 1 - s\log\big(1 - F(x)\big), \qquad s = \frac{1}{\log\left(\frac{x}{\theta}\right)}.$$

that behaves in exactly the same way as the Legendre's spectrum. Thus, monofractal $\lambda(s)$ - spectrum is a cluster of points or a single point while multifractal spectra are single – humped, continuous functions of scale s.

We present some results in relation to the behavior of $\lambda(s)$ in this paper. These results describe the behavior of the $\lambda(s)$ – spectrum when applied to various probability distributions, including the multi – fractal distributions.

2.0 The $\lambda(s)$ – spectrum

Let $x_1, x_2, \ldots, x_n$ be iid $G(.)$ where $G(.)$ is an unknown absolutely continuous distribution with respect to a Lebesque measure. Suppose also that $x_i \geq \theta$ for each i and $\theta > 0$. The idea is to fit a fractal distribution:

(4) $$F(x) = 1 - \left(\frac{x}{\theta}\right)^{1-\lambda}, \quad \lambda > 0, \;\; x \geq \theta$$

to the quantiles $x_{\alpha_1} \leq x_{\alpha_2} \leq \cdots \leq x_{\alpha_n}$ such that $G(x_{\alpha_n}) = \alpha_n$.

Thus,

(5) $$G(x_\alpha) = \alpha = F(x_\alpha) = 1 - \left(\frac{x}{\theta}\right)^{1-\lambda}$$

which reduce to:

(6) $$(1-\lambda)log\left(\frac{x}{\theta}\right) = log(1-\lambda).$$

we obtain:

(7) $$\lambda = 1 - \frac{\log(1-\alpha)}{\log\left(\frac{x}{\theta}\right)}$$

From which the fractal spectral function:

(8) $$\lambda(s) = 1 - \log(1-\alpha)\, s, \qquad s = \frac{1}{\log\left(\frac{x}{\theta}\right)}$$

is obtained.

The fractal spectrum (7) was shown to be a one – to – one function, monotonically increasing with x on a logarithmic scale for non – fractal distributions. Instead of examining the observations on the data space, we propose to examine them in the spectral space. The value of θ used in (8) serves as a powerful "microscope" that enhances the detailed picture of the spectrum $\lambda(s)$ in terms of its finer structures.

We note that if x comes from fractal distribution with fractal dimension λ, then $s = \frac{1}{\log\left(\frac{x}{\theta}\right)}$ decreases with increasing x and decreasing θ. For a fixed observation x, we can increase (decrease) by decreasing θ (increasing θ), so that the value of θ serves to sharper the focus on the features of a fractal set. Viewed on a large scale, monofractal distributions have singular spectra $(P(\lambda(x) = \lambda_0) = 1)$ but when viewed on a lower scale, the spectral function forms a horizontal line (slope = 0). Similarly, multifractal distributions viewed on a larger scale (s) have a spectrum that behaves like power function As^{-k} but when viewed on a smaller scale, it behaves like a concave downward quadratic function $A(s-s_0)^2 + B(s-s_0) + C$ where A, B and C are parameters to be estimated.

3.0 Results

We now state some major results which we found by extensive simulation but which we will now prove mathematically.

Result 1. Let $f_1(x,\lambda_1), f_2(x,\lambda_2), f_3(x,\lambda_3), \ldots f_m(x,\lambda_m)$ be monofractal densities and suppose:

$$f(x) = \sum_{i=1}^{m} w_i f_i(x,\lambda_i), \quad \sum_{i=1}^{m} w_i = 1, w_i \geq 0 \; for\ all\ i$$

is a mixture of monofractals, then $F(x)$ can be expressed as a monofractal with dimension λ:

$$F(x) = 1 - \left(\frac{x}{\theta}\right)^{1-\lambda}$$

where λ is a weighted harmonic mean of λ_i.

Proof: Let

$$F(x) = 1 - \sum_{i=1}^{m} w_i \left(\frac{x}{\theta}\right)^{1-\lambda_i} \quad , \quad \sum_{i=1}^{m} w_i = 1$$

be the cumulative distribution function of the mixture densities *f(x)*. When the observations are taken as a monofractal with dimension λ , then:

$$F(x) = 1 - \left(\frac{x}{\theta}\right)^{1-\lambda}$$

Let $S_* = \frac{x}{\theta}$, then:

$$F(x) = 1 - \sum_{i=1}^{m} w_i \, s_*^{1-\lambda_i} = 1 - (s_*)^{1-\lambda}$$

That is, we wish to find a λ such that:

$$(9) \qquad \frac{w_1}{s_*^{\lambda_1}} + \frac{w_2}{s_*^{\lambda_2}} + \cdots + \frac{w_m}{s_*^{\lambda_m}} = \frac{1}{s_*^{\lambda}}$$

or:

$$w_1\left(s^{\lambda-\lambda_1}\right) + w_2\left(s^{\lambda-\lambda_2}\right) + \cdots + w_m\left(s^{\lambda-\lambda_m}\right) = 1, \ \ \sum_{i=1}^{m} w_i = 1, w_i \geq 0$$

If $w_1 = w_2 = \cdots = w_m = \frac{1}{m}$

$$(10.) \qquad s^{\lambda} = \left[\frac{1}{m}\sum_{i=1}^{m}\frac{1}{s^{\lambda_i}}\right]^{-1} = H$$

is the harmonic mean. It follows that $s^{\lambda} = H$ so that $\lambda = \frac{\log H}{\log s}$. With unequal weights $w_i, \lambda = \frac{\log WH}{\log s}$ where WH is the weighted harmonic mean. ■

Note, however, that the monofractal representation of an inherently multifractal spectra down not have the same spectrum as the original set. This is because the set $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m)$ is just one of infinite number of possibilities to generate a fractal dimension λ as in (10).

Result 2. The probability distribution of the fractal dimension of fractal variables obeys the distribution:

$$g(y) = \frac{(\lambda-1)}{(y-1)^2} e^{-\frac{(\lambda-1)}{y-1}}$$

If $s = \frac{1}{log\left(\frac{x}{\theta}\right)}$, then $g(y) = (\lambda-1)s^2 e^{-(\lambda-1)s}, 1 < y < \infty, \ \ s \geq 0.$

Proof. The maximum likelihood estimator of λ obtained from:

$$f(x) = \frac{\lambda-1}{\theta}\left(\frac{x}{\theta}\right)^{-\lambda} \quad , x \geq \theta, \ \ \lambda > 1$$

is:

$$\lambda = 1 + \left[log\left(\frac{x}{\theta}\right)\right]^{-1}.$$

Let $y = \lambda$, then:

$$G(y) = P(\lambda \leq y) = P\left(1 + \frac{1}{log\left(\frac{x}{\theta}\right)} \leq y\right) = P\left(x \geq \theta \, e^{\frac{1}{y-1}}\right)$$

Hence:

$$G(y) = 1 - F\left(\theta \, e^{\frac{1}{y-1}}\right) = \left(e^{-\frac{(\lambda-1)}{y-1}}\right), \quad 1 < y < \infty$$

and:

$$g(y) = \frac{(\lambda-1)}{(y-1)^2} e^{-\frac{(\lambda-1)}{y-1}} \quad , 1 < y < \infty.$$

as desired ■

We observe that this result is consistent with the behavior of the Legendre spectrum as well. (Lapenna, Macchiato et al (2003)).

Result 3. The information contributed by x on λ is:

$$I(\lambda) = \frac{1}{(\lambda-1)^2} \quad ,$$

where *I(λ) i*s the Fisher's information number.

Proof:

$$I(\lambda) = E\left(\frac{\partial^2 \log f}{\partial \lambda^2}\right) = E\left(\frac{\partial \log f}{\partial \lambda}\right)^2$$

Using:

$$f(x) = \frac{\lambda-1}{\theta}\left(\frac{x}{\theta}\right)^{-\lambda} \qquad ,\lambda > 1,\ x \geq \theta.$$

We obtain:

$$I\lambda = \frac{1}{(\lambda-1)^2} \text{ as desired.} \qquad \blacksquare$$

Result 3 implies that the smaller values of x contribute more to the fractal dimension λ than the larger values. The smaller values, in fact, define the characteristic irregularity and ruggedness of the fractal observations. Larger values of λ, thus, denote more rugged and more irregular fluctuations of the values of x.

A corollary result that pertains to the Fisher's information index is given in result 4.

Result 4. The multifractal spectrum:

$$\lambda(x) = 1 - \frac{\log(a-\alpha)}{\log\left(\frac{x}{\theta}\right)}\ , x \geq \theta$$

where $\alpha = F(x)$ is a bounded function of x.

Proof: We observe what happens to $\lambda(x)$ at the endpoints:

$$\lim_{x\to\infty} \lambda(x) = 1 - \lim_{x\to\infty} \frac{\log(1-\alpha)}{\log\left(\frac{x}{\theta}\right)}$$

The second term is an indeterminate of the form $\frac{\infty}{\infty}$.

Applying L' Hopital's Rule

$$\lim_{x\to\infty} \frac{\log(a-\alpha)}{\log\left(\frac{x}{\theta}\right)} = \lim_{x\to\infty} \frac{\frac{-f(x)}{1-F(x)}}{\frac{1}{x}} = \lim_{x\to\infty} -f(x)\frac{x}{1-F(x)}$$

which is again an indeterminate. Applying L'Hopital Rule again, we obtain:

$$\lim_{x\to\infty} \frac{(1-F(x))}{\log\left(\frac{x}{\theta}\right)} = \lim_{x\to\infty} -f(x)\frac{-xf(x)}{1-F(x)}$$

In the case of monofractals,

$$f(x) = \left(\frac{\lambda-1}{\theta}\right)\left(\frac{x}{\theta}\right)^{-\lambda}$$

and

$$F(x) = 1 - \left(\frac{x}{\theta}\right)^{-\lambda} \qquad ,1 - F(x) = \left(\frac{x}{\theta}\right)^{-\lambda}$$

Hence:

$$\lim_{x\to\infty} \frac{log(1-F(x))}{\log\left(\frac{x}{\theta}\right)} = \lim_{x\to\infty} -x\frac{\lambda-1}{x} = -(\lambda-1) = 1-\lambda$$

It follows that

$$\lim_{x\to\infty} \lambda(x) = 1 - (1-\lambda) = \lambda.$$

At the other end, where $\to \theta$, the minimum, we obtain:

$$\lim_{x\to\theta} \frac{log(1-F(x))}{\log\left(\frac{x}{\theta}\right)}$$

which is an indeterminate of the form $\frac{0}{\infty}$.

Applying L'Hopital Rule:

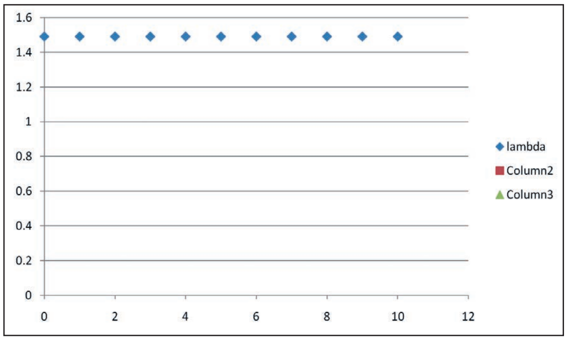
$$\lim_{x\to\theta} \frac{log(1-F(x))}{\log\left(\frac{x}{\theta}\right)} = \lim_{x\to\theta} \frac{-x\,f(x)}{1-F(x)} = \frac{-\theta\,f(\theta)}{1-F(\theta)} = 1-\lambda$$

Thus:

$$\lim_{x\to\theta} \lambda(x) = 1-\lambda \text{ also.}$$

We then conclude that $\lambda(x) = \lambda$ at the extreme endpoint of x, and is therefore bounded for monofractal observations.

The graph of the λ(s) spectrum for typical monofractals is shown below:



In the case of multiple fractional dimensions or multifractals, we can represent the multifractal density as:

$$f(x) = \sum_{i=1}^{m} w_i f_i(x, \lambda_i),$$

as found in Result 1. The corresponding cumulative distribution function is:

$$F(x) = 1 - \sum_{i=1}^{m} w_i \left(\frac{x}{\theta}\right)^{1-\lambda_i} \quad , \quad \sum_{i=1}^{m} w_i = 1$$

The corresponding values at the endpoints are:

As x →∞ or s →0, then,

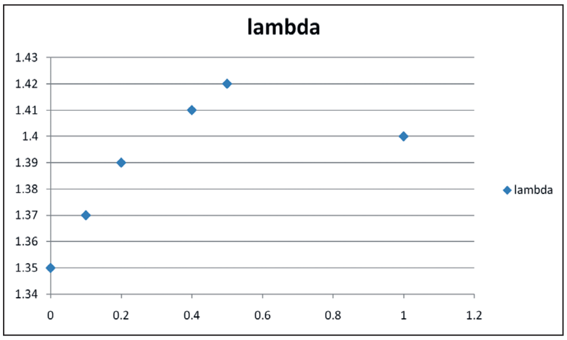
$\lim_{x\to\infty} \lambda(x) = \lambda_1$, λ_1 is the minimum of the λ's,

while at the other extreme point x →θ or s →∞

$\lim_{x\to\theta} \lambda(x) = \sum_{i=1}^{m} w_i \lambda_i$

which is again a bounded function. ■

A typical multifractal λ(s) spectrum is shown below:



Note that the multifractal spectrum is a single-humped continuous function of scale (s). This was also mentioned in the work of Lapenna et al. (2003) using the Legendre spectrum method.

For random variables not distributed as monofractal, the behavior of the spectrum is quite different. In such cases, the multifractal spectrum monotically increases with x the rate of increase appears to involve a power law.

Result 5. Let x be distributed according to some exponential distribution belonging to class:

$$f(x) = Ae^{-h(x)} \qquad ,x > 0$$

The multifractal spectrum $\lambda(x)$ obeys a power – law pattern provided $h(x)$ is a polynomial in x.

Proof:

Let $\lambda(x) = 1 - \frac{log(1-F(x))}{\log\left(\frac{x}{\theta}\right)}$

And we examine the behavior of the $\lambda(x)$ as $x \to \infty$ and as $x \to 0$.

From:

$$\lim_{x\to\theta} \frac{log(1-F(x))}{\log\left(\frac{x}{\theta}\right)} = \lim_{x\to\theta}\left(1 + \frac{f(x)}{f(x)}\right)$$

We obtain:

$$\lim_{x\to\theta}\frac{log(1-F(x))}{\log\left(\frac{x}{\theta}\right)}=1-\theta h(\theta)$$

Thus,

$\lambda(x)\to\theta h(\theta)$ as $x\to\theta$. Now if the $\deg(h(x))=p$, then $\deg(h(\theta))=p-1$ so that $\lambda(x)\to p$ degree polynomial in θ as required. ■

Obviously $\lambda(x)\to\infty$ as $x\to\infty$, hence $\lambda(x)$ is a monotonic increasing polynomial in x. Result 5 shows that if x is distributed as , then $abs(N(0,1))$, then $h(x)=\frac{1}{2}x^2$ and $\lambda(x)\cong\theta^2$ near $x=\theta$; if x distributed as $exp\ (\beta)$, then $h(x)=\beta x$ so $\lambda(x)\cong\beta\theta$ near $x=\theta$.

Work in Progress

We suspect a strong connection between multifractal analysis and the density of primes problem as embedded in the Riemann hypothesis for prime numbers. For instance, the density of primes less than or equal to a real number x is asymptotically equal to 1/log(x) which happens to be the scale in our proposed multifractal spectrum. We are currently working on the problem using the known number of primes up to x = 100,000,000.

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