On Fractional and Fractal Derivatives in Relation to the Physics of Fractals

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Abstract

Fractional and fractal derivatives are both generalizations of the usual derivatives that consider derivatives of non-integer orders. Interest in these generalizations has been triggered by a resurgence of clamor to develop a mathematical tool to describe "roughness" in the spirit of Mandelbrot's (1967) fractal geometry. Fractional derivatives take the analytic approach towards developing a rational order derivative while fractal derivatives follow a more concrete, albeit geometric approach to the same end. Since both approaches alleged to extend whole derivatives to rational derivatives, it is not surprising that confusion will arise over which generalization to use in practice. This paper attempts to highlight the connection between the various generalizations to fractional and fractal derivatives with the end-in-view of making these concepts useful in various physics applications and to resolve some of the confusion that arise out of the fundamental philosophical differences in the derivation of fractional derivatives (non-local concept) and fractal derivatives (local concept).

Keywords: fractional derivative, fractal derivative, fractional differential operators, fractal analysis

1.0 Introduction

The origins of fractional calculus can be traced back to a 1695 paper by L'Hospital to Leibnitz when he queried on the possibility of a fractional derivative and a possible meaning of "half derivative" of x. Leibnitz responded by saying that this could lead to a paradox for which the consequences can be useful in the future. The recent flurry of research in the field of fractal analysis, particularly in the last decade, appears to be the "future" alluded to by Leibnitz in his response. (Mandelbrot (1967), Palmer (1987), Selvam (2007))

The paradox of a fractional derivative can be attributed to the fact that , indeed, there are a variety of ways of generalizing the concept of a differential operator raised to rational powers which result in non-equivalent answers. Thus, a "half derivative" can be obtained using one generalization which will turn out to be a different "half derivative" when obtained using yet another generalization. This paper explores the various generalization of the concept of differential operators raised to rational powers and provides some insights into their possible application in fractal analysis of physical phenomena. We mention that the paper is motivated by the comments of an external referee to the paper "On Fractional Derivatives " (Borres et al. (2013)) which essentially pointed out the need for a thorough exposition on the various ways in which fractional calculus can be developed.

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2.0 Generalization to Fractional Derivatives Based on Exponential and Polynomial Functions

Consider the exponential function:

1. f(x) = eax

whose kth derivative is:

2. $\frac{d^k}{dx^k} e^{ax} = a^k e^{ax}$.

A natural way to generalize to rational derivatives is to set k = q ε R+ to yield:

3. $\frac{d^q}{dx^q} e^{ax} = a^q e^{ax}$.

Any function that can be expressed as a linear sum of exponential functions can be differentiated in rational orders. In particular, since:

4. $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

It follows that:

5. $\frac{d^q}{dx^q} (\cos(x)) = \cos\left(x + \frac{\pi}{2} q\right)$.

The last result indicates that the fractional differential operator shifts the phase of the trigonometric function in fractional proportion. The same statement can be said of the sine function. The approach using exponential functions as a way to generalize to fractional derivatives appears to be satisfactory yet, L'Hospital wanted the "half derivative" of $f(x) = x$. There is no apparent representation of $f(x)$ as an exponential function taking the form of (1). Note that f(x) = elog(x) is not the same as (1). The other way is to find the Fourier representation of the function (which is not defined over the entire interval) over some finite interval. However, ambiguities cannot be avoided here because of the choice of the finite interval. In other words, the fractional derivative obtained in this manner is neither unique nor local (i.e. whole derivatives are both unique and local).

Instead of beginning with an exponential function, we can attempt to build fractional derivatives by considering the polynomial function:

6. $f(x) = x^m$

whose kth derivative is :

7. $\frac{d^k}{dx^k} x^m = m(m-1)(m-2)\ldots(m-k+1)x^{m-k}$

The Gamma function or generalized factorial can be used to make the result above applicable to any positive rational order of derivative. The Gamma function is :

8. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx = (\alpha-1)\Gamma(\alpha-1)$

, which allows us to write (7) as:

9. $\frac{d^k}{dx^k} x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-k)} x^{m-k}$

with the recurrence relation $\Gamma(n) = (n-1)!$ and the reflection relation $\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}$

Replacing k with q ε Q+, we obtain the desired generalization. In response to Leibnitz, the half derivative of x now becomes:

10. $\frac{d^{1/2}}{dx^{1/2}} (x) = \frac{2}{\sqrt{\pi}} x^{1/2}$

since $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

While we are used to thinking of the derivative of a constant to be equal to zero, the half-derivative of a constant is not zero but:

11. $\frac{d^{1/2}}{dx^{1/2}}(c) = \frac{c}{\Gamma(1/2)}x^{-1/2} = \frac{c}{\sqrt{\pi x}},\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

which is not defined at x = 0.

We point out that combining the two approaches lead to some curious results. For instance, we know that:

12. $e^x = \sum_0^\infty \frac{x^n}{n!}$

but

13. $\frac{d^{1/2}}{dx^{1/2}} e^x = (1)^{1/2}\ e^x = e^x$

$$\frac{d^{1/2}e^x}{dx^{1/2}}(e^x) = \frac{d^{1/2}}{dx^{1/2}}\sum_0^\infty \frac{\Gamma x^n}{n!} = \sum_0^\infty \frac{\Gamma(n+1)x^{n-1/2}}{\Gamma(n+\frac{1}{2})} \neq e^x$$

Results such as (13) are not rare and for which reason, Leibnitz was right in pointing out that generalization to rational order derivatives indeed lead to paradoxes.

3.0 Generalization to Fractional Derivatives Based on a Limit Definition

We next examine yet another approach that is more basic than the derivations of fractional derivatives based on the exponential and polynomial functions. The definition of a positive integer derivative is based on:

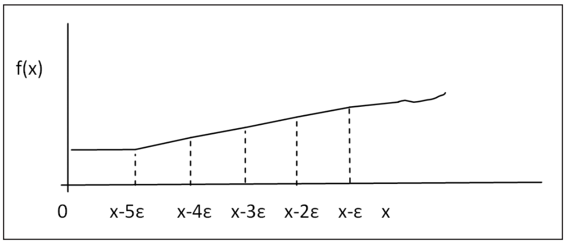
14. $\frac{d}{dx}\left(f(x)\right) = \lim_{\varepsilon\to 0}\frac{f(x)-f(x-\varepsilon)}{\varepsilon}$

which can be repeated n times to yield:

$$\frac{d^n}{dx^n}\left(f(x)\right) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\sum_{j=0}^{n}(-1)^j\, nC_j\ f(x-j\varepsilon)$$

where nCj is the number of combinations of n objects taken j at a time. In order to generalize this formula when n is not a positive integer, we need to generalize in two directions: the binomial coefficients and the upper limit of the summation.

Consider a function f(x) shown below :



Given ε > 0, we move five equi-steps to the left to obtain the points x - 5ε, x - 4ε, ..., x – ε.. In general, let k be the number of such points , then x = kε. Define the ε- backward shift operator as: Bε(f(x) = f(x-ε). The general differential operator D is defined by:

16. $D^n[f(x)] = \lim_{\varepsilon\to 0}\left(\frac{(1-B_\varepsilon)^n}{\varepsilon^n}\right) f(x)$

For n = 1, we obtain the usual differential operator:

17. $D\left(f(x)\right) = \lim_{\varepsilon\to 0}\frac{f(x)-f(x-\varepsilon)}{\varepsilon}$

while for n = -1, we make use of the geometric series representation:

18. $\frac{\varepsilon}{(1-B_\varepsilon)} = \varepsilon[1 + B_\varepsilon + \ldots + {B_\varepsilon}^k + \ldots]$

This gives:

19. $D^{-1}[f(x)] = \lim_{\varepsilon\to 0}\left(\frac{(1-B_\varepsilon)^{-1}}{\varepsilon^{-1}}\right) f(x)$

$= \varepsilon[\, f(x) + f(x-\varepsilon) + f(x-2\varepsilon) + \cdots f(x-k\varepsilon) + \ldots]$

Equation (16) gives the differential operator while (19) yields the integration operator in the limit as ε tends to zero. An algebraic expression for the Fundamental Theorem in Calculus can be obtained from (19) and (9) namely, that:

"If $\frac{d^k}{dx^k}x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-k)}x^{m-k} = D^k x^m$, then for k=-1, $D^{-1}(x^m)$ is the integral of x^m"

For example, if $f(x) = x^2$, then $D^{-1}(x^2) =$

$$\frac{2!}{\Gamma(2+1-(-1))} x^{2-(-1)} = \frac{2!}{3!} x^3 = \frac{1}{3} x^3.$$

The upper limit of the summation in Equation (15) becomes:

$$\frac{d^n}{dx^n}(f(x)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \sum_{j=0}^{\left[\frac{x-x_0}{\varepsilon}\right]} (-1)^j \, nC_j \, f(x - j\varepsilon)$$

where x_0 is usually zero, or $\left[\frac{x}{\varepsilon}\right]$

We then proceed to define the binomial coefficients in a way that does not require integer values of n. Note that:

$$nC_j = \frac{n!}{j!(n-j)!}$$

which suggests that an alternative way of writing factorials for non-integer arguments be used. This is facilitated by using the Gamma function (8). Doing so, we obtain:

20. $$\frac{d^q}{dx^q}(f(x)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^q} \sum_{j=0}^{\left[\frac{x-x_0}{\varepsilon}\right]} (-1)^j \frac{\Gamma(q+1)}{j!\,\Gamma(q+1-j)} f(x - j\varepsilon)$$

The general derivative depends on the value of the function f over the range from x0 to x and is therefore a non-local operation. We can see this from the fact that in the factor f(x-jε) in the summation, the argument ranges from x to zero as j ranges from j = 0 to j =[(x-x0)/ε].

Using (20), L'Hospital's query to Leibnitz yields:

21. $$\frac{d^{1/2}}{dx^{1/2}}(x) = \frac{2}{\sqrt{\pi}} x^{1/2}$$

which is identical with (8).

We therefore see that the generalized derivative obtained by the polynomial approach and that which is obtained by the limit approach give identical results. The utility of using the generalized limit approach over the polynomial approach is that the generalized limit approach allows us to find rational order derivatives even if the function is not expressible as a power series. Moreover, the polynomial approach does not give us any indication about the non-locality of the generalized derivative, that is, the dependence on the function over a range rather than on just a single point.

4.0 Reconciling the Exponential Approach and the Generalized Limit Approach to Fractional Derivatives

In Section 2.0, we pointed out the conflicting results when we took the half-derivative of ex using the exponential approach and the polynomial approach (or the generalized limit approach). The derivative of this function (including half derivatives) should yield the same function. Moreover, the function approaches 1 as x tends to zero. Yet , the half derivative of ex is:

22. $$\frac{d^{1/2}}{dx^{1/2}}(f(x)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{1/2}} \sum_{j=0}^{\left[\frac{x-x_0}{\varepsilon}\right]} (-1)^j \frac{\Gamma(3/2)}{j!\,\Gamma(3/2-j)} f(x - j\varepsilon)$$

$$= \frac{1}{\sqrt{\pi x}}\left(1 + 2x + \frac{4}{3}x^2 + \frac{8}{15}x^3 + \frac{16}{105}x^4 + \cdots\right.$$

which goes to infinity as x goes to zero. How can we reconcile the two results? The answer lies on the fact that half-derivatives depend on the chosen range of differentiation. For instance,

$$\int_{x_0}^{x} e^x = e^x - e^{x_0}$$

and the only way for the derivative of the right–hand side to equal to the argument of the definite integral on the left-hand side is when $x_0 = -\infty$ which we have assumed tacitly. In other words, ranges of differentiation are important when we talk of fractional derivatives (derivatives over a range of values rather than derivative at a single point). If $x_0 = -\infty$, then we find that the upper limit of (22) simply becomes ∞ and hence, the half-derivative in (21) becomes ex also.

5.0 Fractal Dimensions, Roughness, Fractal Calculus and Fractional Calculus

The motivation for developing Fractional Calculus stemmed from a purely analytic question of L'Hospital (1695). Recent interest in fractal geometry and analysis, however, spawned yet another direction for generalizing the usual derivative. This direction is towards defining Fractal Derivatives as basis for describing data and geometric roughness. As such, we expect a relationship between fractal derivatives and fractal dimensions. What is not clear yet is the relationship between fractal derivatives and fractional derivatives.

Fractal Derivative

Let f(t) be a function. We define the qth fractal derivative of f(t) with respect to a fractal measure t as:

23. $$\frac{d\,f(t)}{dt^q} = \lim_{t_1 \to t} \frac{f(t_1) - f(t)}{t_1{}^q - t^q}, q > 0$$

Note at once that while for fractional derivatives $\varepsilon = |t - t_1|$ for fractal derivatives we have $\varepsilon^q = |t_1{}^q - t^q|$. This implies that

24 $$\frac{d\,f(t)}{dt^q} = \lim_{\varepsilon \to 0} \frac{f(t_1) - f(t_1^q - \varepsilon^q)^{1/q}}{\varepsilon^q}$$

Comparing (24) and (20), we see that (24) cannot be expressed in terms of simple backshift operators. In particular, the half – derivative of $f(t) = t$ becomes:

25. $$\frac{df(t)}{dt^{\frac{1}{2}}} = 2\sqrt{t}$$

which differs from (21) by a factor of $\frac{1}{\sqrt{\pi}}$. A more general representation of a fractal derivative with respect to both f(t) and t is:

26. $$\frac{df^r}{dt^q} = \lim_{t_1 \to t} \frac{f^r(t_1) - f^r(t)}{t_1{}^q - t^q}, r, q > 0.$$

Hence for f(t) = t with $r = q = \frac{1}{2}$:

27. $$\frac{d\,f^{1/2}}{dt^{\frac{1}{2}}} = \lim_{t_1 \to t} \frac{\sqrt{t_1} - \sqrt{t}}{\sqrt{t_1} - \sqrt{t}} = 1$$

which coincides with $\frac{df}{dt} = 1$.

Observations:

The fractal derivative $\frac{df}{dt^q}$ is always a function of the fractional derivative $\frac{d^q f}{dt^q}$.

To avoid confusion, we change notation for fractal derivatives. Instead of writing $\frac{df^r}{dt^q}$, we write $G(r, q)$ for Equation (26). Likewise, instead of writing $\frac{d^q f(t)}{dt^q}$ for the qth fractional derivative, we write A(q). The $G(.,.)$ notation reflects the geometric nature of (26) while the $A(.)$ notation highlights the analytic feature of the fractional derivative Equation (20)

Statement (a) can be written as:

$$A(q) = T\big(G(0, q)\big),$$

where T(.) is a functional of $G(.,.)$.

We can compare the fractal derivative $G(0, \frac{1}{2})$ of $f(t) = e^t$ with its corresponding fractional derivative $A\left(\frac{1}{2}\right)$. For reference, we reproduce (22) below:

$$A\left(\frac{1}{2}\right)[e^t] = \frac{1}{\sqrt{\pi}}\left(t^{-\frac{1}{2}} + 2t^{\frac{1}{2}} + \frac{3}{4}t^{\frac{3}{2}} + \frac{8}{15}t^{\frac{5}{2}} + \frac{16}{105}t^{\frac{7}{2}} \ldots +\right)$$

$\rightarrow$ *Fractional Derivative*

The corresponding fractal derivative is:

$$G\left(0,\tfrac{1}{2}\right)[e^t] = 2t^{\frac{1}{2}} + 2t^{\frac{3}{2}} + \tfrac{1}{3}t^{\frac{5}{2}} + \tfrac{1}{3}t^{\frac{7}{2}} \ldots + \rightarrow \textit{Fractal Derivative}$$

For $t > 0$, the fractal derivative is generally greater than the fractional derivative. As $t > 0$, the fractional derivative tends to infinity while the fractal derivative tends to zero; both behaviors being inconsistent with the fractional derivative of e^t using the exponential approach viz. f $f^{\left(\frac{1}{2}\right)}(e^t) = (1)^{\frac{1}{2}}e^t = e^t \rightarrow 1$ as $t \rightarrow 0$. The behaviors of the derivatives $A\left(\frac{1}{2}\right)$, $G\left(0,\frac{1}{2}\right)$ and $f^t(t)$ of e^t are shown below

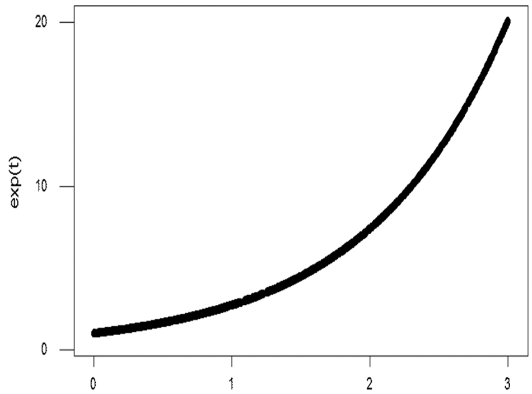


Figure 1: e^t

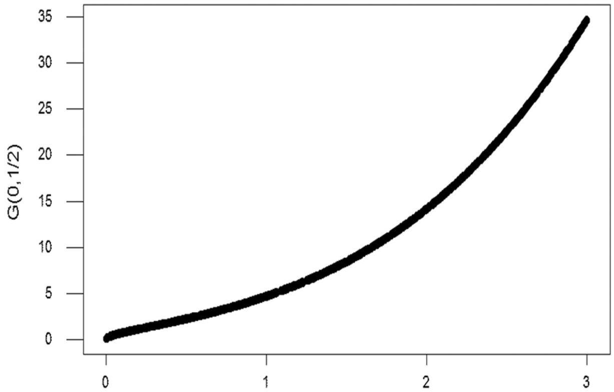


Figure 2: half fractal derivative

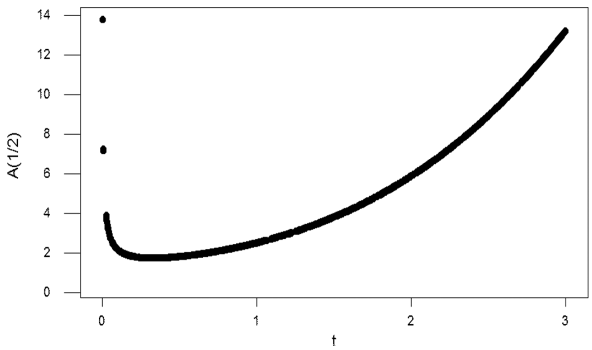


Figure 3: half fractional derivative

The tables below indicate the values of the half derivatives

0 < t < .51

t	exp(t)	A(1/2)	G(0,1/2)
0.00169	1.0017	13.77046	0.082358
0.00622	1.0062	7.242892	0.158716
0.00636	1.0064	7.164721	0.160515
0.02289	1.0232	3.901294	0.309542
0.0237	1.024	3.84009	0.315223
0.02547	1.0258	3.717008	0.327352
0.02599	1.0263	3.683348	0.330845
0.02788	1.0283	3.569347	0.343301
0.03151	1.032	3.381066	0.366268
0.50395	1.6552	1.80922	2.225664

1 < t < 1.036

t	exp(t)	A(1/2)	G(0,1/2)
1.00019	2.7188	2.502952	4.66774
1.00045	2.7195	2.503464	4.6693
1.00645	2.7359	2.515332	4.705406
1.01012	2.7459	2.522633	4.727583
1.01249	2.7524	2.527365	4.741942
1.01273	2.7531	2.527845	4.743398
1.01772	2.7669	2.537853	4.773732
1.02446	2.7856	2.551465	4.814913
1.02761	2.7944	2.557863	4.834241
1.03589	2.8176	2.574793	4.885299

1.5 < t <1.55

T	exp(t)	A(1/2)	G(0,1/2)
1.50904	4.5224	3.83446	8.503707
1.5178	4.5622	3.863687	8.585542
1.51835	4.5647	3.86553	8.590701
1.51986	4.5716	3.870594	8.604875
1.52179	4.5804	3.877077	8.623018
1.52312	4.5865	3.881551	8.635538
1.52507	4.5955	3.88812	8.653921
1.52573	4.5985	3.890347	8.660149
1.52706	4.6046	3.894837	8.672711
1.53064	4.6211	3.90695	8.706595

2 < t < 2.05

t	exp(t)	A(1/2)	G(0,1/2)
2.00068	7.3941	5.86979	14.15103
2.01024	7.4651	5.918051	14.28454
2.01281	7.4843	5.931086	14.3206
2.0134	7.4887	5.934082	14.32889
2.01531	7.503	5.943791	14.35575
2.03163	7.6265	6.02734	14.58689
2.03406	7.6451	6.03987	14.62156
2.03733	7.6701	6.05677	14.66831
2.04045	7.6941	6.072934	14.71304
2.04153	7.7024	6.078538	14.72854

In the neighborhood of zero, the analytic fractional derivative is larger than the geometric fractal derivative. As we move away from zero (somewhere near t = 0.50), the geometric fractal derivative dominates both the analytic fractional derivative and the fractional derivative obtained through the exponential approach. There appears to be some simple relationship that governs the behavior of the exponential fractional derivative (e^t) and the geometric fractal derivative. We performed a regression analysis using $y = e^t$ and $x = G(0,1/2)$. The results are shown below:

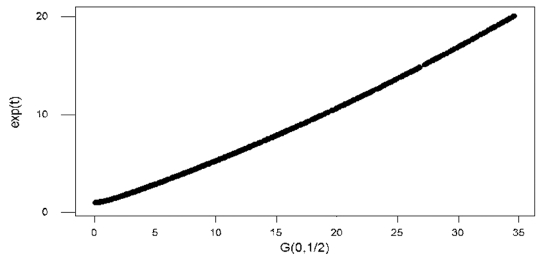


Figure 4: Graph of x = G(0,1/2) versus y = exp(t)

The regression equation is
exp(t) = 0.198 + 0.539 G(0,1/2)

Predictor	Coef	SE Coef	T	P
Constant	0.19798	0.01841	10.75	0.000
G(0,1/2)	0.539015	0.001241	434.33	0.000

S = 0.3696   R-Sq = 99.5%   R-Sq(adj) = 99.5%

That is, the exponential fractional derivative is about one-half of the geometric fractal derivative. The two plots of the other possible relationships between the three half-derivatives are shown below:

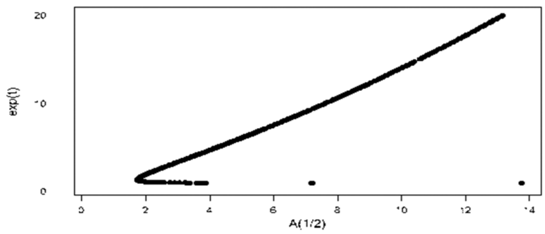


Figure 5: Plot of x = A(1/2) and y = exp(t)

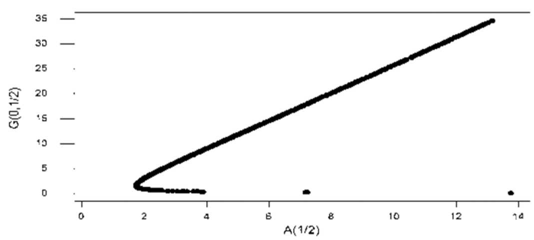


Figure 6: Plot of x = A(1/2) and y = G(0,1/2)

Note that Figures 5 and 6 are almost identical. This observation simply comes from the fact that the geometric fractal derivative (G(0,1/2)) and the fractional derivative obtained by the exponential approach (et) are linearly related.

6.0 Some Application of Fractal Derivatives

Physical laws of diffusion that are based on a Euclidean medium cannot apply if in fact the media exhibit fractal properties. Some physical realizations of fractal media include water turbulence porous media and aquifers. In the latter case, the physical concepts of space and time have to be rescaled to , (xr, tq) where r, q are positive real numbers, in order to conform to the fractal nature of space-time. In this fractal space- time system, the notion of a velocity can be redefined as:

28. $G(r,q)S(x,t) = \frac{dx^r}{dt^q}$

because the traditional definition of velocity is meaningless in a non-differentiable fractal space-time.

Fractal Linear Motion

Linear motions involving the concepts of velocity and acceleration are fundamental in Physics. The usual distance traversed by a free-falling body after time t is:

29. $x(t) = \frac{1}{2}gt^2$

where g is assumed constant. Suppose that the space $x(t)$ is fractal of dimension r, then the velocity of the body at any constant t, becomes

30. $\frac{d(x^r)}{dt} = \lim_{t\to t_1} \frac{\frac{1}{2}gt^{2r} - (\frac{1}{2}gt_1^{\,2})^r}{t-t_1}$

$= \frac{-r}{2^{r-1}} g^r t_1^{\,2r-1}$

Note that if the fractal dimension of $x(t)$ is $r = 1,$ we obtain the usual velocity as:

$v = -gt.$

However, if $x(t)$ is fragmented with, say, $r = 0.63,$ the velocity is:

$$v^{0.63} = -(2^{0.37} x(0.63) g^{0.63} t_1^{\ 0.26})$$

which is slower than gt, as one would expect if one were travelling along a rough media e.g variable air resistance.

Fick's Second Law Anomalous Motion

Consider the anomalous transport-diffusion process which is an alternative to Fick's Second Law given by:

31. $\frac{du(x,t)}{dt^q} = D(\frac{\partial}{\partial x^r}\left(\frac{\partial u(x,t)}{\partial x^r}\right), -\infty < x < \infty, u(x,0) = \delta(x)$

where δ(x) is the Dirac delta function, and 0 < q < 2, 0 < r < 1. The fundamental solution to (32) can be obtained by the transformation t* = tq and x* = x^r which transform (32) into the usual diffusion equation. The solution , of course, is given by:

32. $u(x,t) = \frac{1}{2\sqrt{\pi t^q}} \exp(\frac{-x^{2r}}{4t^q})$

which is the stretched normal (Gaussian) distribution. From this example, we learn that in order to deal with fractal space-time, it is necessary to know the fractal dimensions r and q of the space (x) and the time (t) respectively. The corresponding fractal derivatives can be calculated and the solution to a fractal differential equation obtained by transformation of variables.

Fourier's Law of Heat Conduction

As a third example, consider Fourier's law (1822) (see Fourier, (1955)) which states that "thermal conduction results into heat flux which is proportional to the magnitude of the temperature gradient and opposite to it in sign". Thus, for a one-direction conduction process, this can be translated to:

33. $h_x = -c\frac{dT}{dx}$

where c is the thermal conductivity constant, h_x is the heat flux in W/m^2 in the positive x direction and dT/dx is the (negative) temperature gradient (K/m) in the direction of the heat flow (usually, from hot to cold regions). Suppose now that the space (x) is fractal of dimension r while T is of fractal dimension q, then Fourier's law translates to:

34. $h_x = -c\frac{dT^q}{dx^r}$

whose solution is obtained by the transformation x* = x^r and T* = T^q. Equation (34) is the anomalous heat conduction equation.

Other examples may be constructed where the use of fractal derivatives is examplified. The reader may wish to refer to Darcy's law. Darcy's law relates the instantaneous discharge rate through a porous medium and the viscosity of the fluid with pressure drop over a given distance.

7.0 Conclusion

While we have found a linear relationship between fractal derivatives (G(0,q)) and fractional derivatives through the exponential definition, much is still to be done in terms of relating analytic fractional derivatives (A(q)) and fractal derivatives (G(0,q)). A relationship between A(q) and G(0,q), if found, would be useful in terms of developing an entire theory of fractal calculus (as opposed to fractional calculus). A fractal calculus theory would, in turn, be useful in re-analyzing most of the laws in Physics e.g. quantum theory. The key to such an examination lies on the premise of a local versus non-local derivatives.

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