On Fractional Derivatives and Application*

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Abstract

In many instances, the first derivative of the functional representation f(x) will fail to exist (Mandelbrot, 1987). When this happens, it is important to develop an appropriate language to describe these minute finer roughness and irregularities of the geometric objects. This paper attempts to develop the calculus of fractional derivatives for this purpose. Local approximations to functional values by fractional derivatives provide finer and better estimate than the global approximations represented by power series e.g Mclaurin's series. Fractional derivatives incorporate information on the fluctuations and irregularities near the true functional values, hence, attaining greater precision.

Keywords: local approximation, global approximation, fractional derivatives

1.0 Introduction

Let $f = D \to R$ be a function from D to R. The first derivative of f with respect to x is defined as:

(1)… $\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

if the limit exists. The first derivative of f is itself a function, denoted by:

(2)… $f(x) = \frac{df}{dx}$.

Since (2) is also a function, it is possible to apply (1) to obtain the second derivative of f with respect to x:

(3)… $\frac{d(f)}{dx} = \frac{d^2 f}{dx^2}$.

The first and second derivatives of a function f have many applications. For instance, if $s = f(t)$ is a function describingthe distance (s) covered by a moving body for time t, then $\frac{ds}{dt} = f'(t)$ and $\frac{d^2 s}{dt^2} = f''(t)$ denote the velocity (speed) and acceleration of the body at any given time t.

Geometrically, $f'(t)$ represents the slope of the tangent line at time t and $f''(t)$ shows its concavity at that point. While the first and the second derivatives of a function are well- understood, it is logical to ask if a fractional derivative, i.e $f^{(8)}(t)$ where q∈ $\mathbf{Q}^+$ in a positive rational number can be formulated. Thus, we may ask for the half-derivatives, $f^{\left(\frac{1}{2}\right)}(t)$, or the one-eight derivative, $f^{\left(\frac{1}{8}\right)}(t)$. This equation arises not only as a purely logical consequence of integer derivatives but quite naturally stems from a study of finer roughness and ruggedness of geometric objects dealt with in the study of fractals (Mandelbrot, 1967). Nature and natural processes are fractals or highly rugged and irregular. In contrast, the mathematical tools developed for describing geometric objects are designed for smoothness. For example, functions that possess derivatives of all orders are smooth function. Consequently, all such mathematical representations fail to properly describe natural geometric objects which are extremely rough e.g. cloud formation, plant leaf

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structures and mountains. In many instances, even the first derivative of the functional representation *f(x)* will fail to exist (Mandelbrot, 1987). When this happens, it is important to develop an appropriate language to describe these minute finer roughness and irregularities of the geometric objects. This paper attempts to develop the calculus of fractional derivatives for this purpose.

2.0 Fractional Derivatives

Consider the monomial:

(4)… $(x) = x^m, m \in Z^+$

Whose kth derivative is:

(5…$f^k(x) = m(m-1)\ldots(m-k+1)x^{m-k}$.

We rewrite expression (5) in terms of the more general gamma function:

(6)…$f^k(x) = \frac{\Gamma(m+1)}{\Gamma(m+1-k)} x^{m-k}$,

where:

(7)… $\Gamma(m) = \int_0^\infty x^{m-1} e^{-x}\, dx$

using (6), it is expedient to define the k^{th} fractional derivative of f(x) as

Definition: Let $f(x) = x^m$ be an m^{th} degree monomial. The k^{th} fractional derivative of f, $k \in \mathbf{Q}^+$, and $k \leq m$ is given by Equation (6):

$$f^k(x) = \frac{\Gamma(m+1)}{\Gamma(m+1-k)} x^{m-k} \quad, 0 < k \leq m$$

$$= 0 \qquad , \text{else.}$$

Note that if k is a positive integer less or equal to m, then the k^{th} fractional derivative reduces to the usual derivative of f. For k=1 the definition gives:

$$f'(x) = \frac{\Gamma(m+1)}{\Gamma(m)} mx^{x-1}$$

as expected. The fractional derivative operator $f^{(q)}(\cdot), q \epsilon Q^+$, obeys the usual linearity rule:

Theorem 1. Let $f(x) = a_0 + a_1 + a_2x^2 + \cdots + a_mx^m$.

Then,

$f^{(q)}(x) = a_0 f^{(q)}(1) + a_1 f^{(q)}(x) + \cdots + a_m f^{(q)}(x)$.

Proof: The result follows directly from the definition. ■

Our interest centers on roughness as represented by the first derivative. More precisely, when the first derivative fails to exist for a large class of functions.

Let $\{q_n\} \uparrow 1$ be a sequence of rational numbers conveying to 1. We examine the behavior of $f^{(q_n)}(x)$ as $n \to \infty$. It is easy to see that:

$$f^{(q_n)}(x) \to f'(x) as n \to \infty$$

We state this as a Theorem.

Theorem 2. Let $f(x) = x^m$ and let $\{q_n\}$be a sequence of rational numbers increasingto 1. Then,

$$f^{(q_n)}(x) \to f(x) as n \to \infty$$

Proof:

From:

$$f^{(q_x)}(x) = \frac{\Gamma(m+1)}{\Gamma(m+1-q_n)} x^{m-q_n} \quad ,$$

We observe that $\Gamma(m+1-q_n) \to \Gamma(m)$ because $q_n \to 1$. Since polynomial functions are continuous, it follows that $x^{m-q_n} \to x^{m-1}$, hence:

$$f^{(q_n)}(x) = \frac{\Gamma(m+1)}{\Gamma(m+1-q_n)} x^{m-q_n} \to \frac{\Gamma(m+1)}{\Gamma(m)} x^{m-1} = m x^{m-1} = f(x) \blacksquare$$

Next, suppose that $q_n \downarrow 0$ as $n \to \infty$, we observe that:

$$f^{(q_n)}(x) = f^{(0)}(x) = f(x)$$

which can be easily deduced from the definition of $f^{(q_n)}(x)$ and the continuity of the polynomial function. The graph of q_n versus $f^{(q_n)}(x)$ for a value of x and $0 < q_n \leq 1$ is shown below:

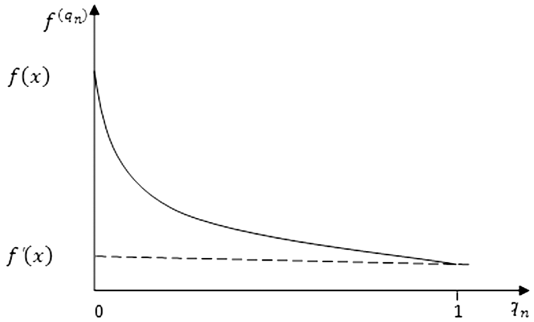

Figure 1. Behavior of $f^{(q_n)}(x), 0 < q_n \leq 1$

Geometric View of the Fractional Derivatives

In order to get an intuitive feel for fractional derivatives, we begin from the well- accepted characterization of the first derivative, the slope of the tangent line to a curve at a point $(x, f(x))$:

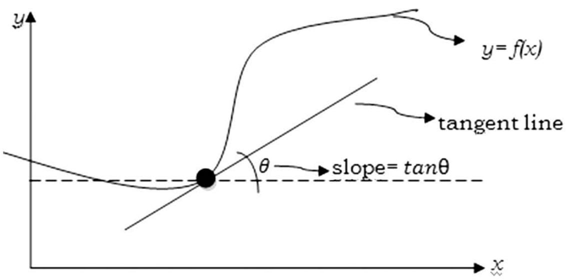

Figure 2. The first derivative, the slope, $\frac{dy}{dx} = tan\theta$

The fractional derivatives $f^{(q_n)}(x)$ for $0 < q_n \leq 1$ represent various slopes as illustrated below:

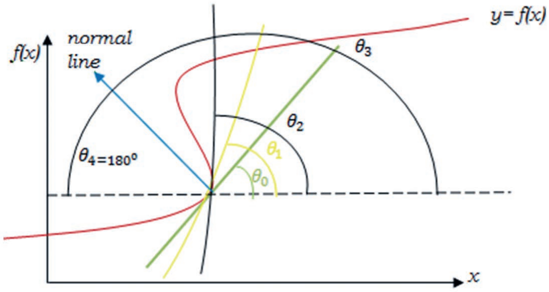

Figure 3. Fractional derivatives, slopes of various lines related to the tangent line

Fractional Derivatives for Arbitrary Differentiable Functions

Let f(x) be a continuously differentiable function which admits a Mclaurin's series expansion of the form:

$$(8)\ldots\ f(x) = f(0) + f'(0)x + \frac{f''(0)\,x^2}{2!} + \frac{f'''(0)\,x^3}{3!} + \ldots$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(0)x^k}{k!} = \sum_{k=0}^{\infty} a_k\, x^k$$

as a specific example, let f(x) = e^x whose Mclaurin's series expansion is given by:

$$(9)\ldots\ f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

For functions of this type, the corresponding fractional derivatives are given by:

Definition 2: Let f(x) admit a Mclaurin's series expansion of the form (8).

Then:

$$f^{(q_n)}(x) = \sum_{k=0}^{\infty} a_k f^{(q_n)}(x^k) \text{ for } 0 < q_n \leq 1$$

3.0 Approximations by Fractal Derivatives

In mathematics, a power series (in one variable) is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c)^1$$
$$+a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

where a_n represents the coefficient of the nth term, c is a constant, and x varies around c. This series usually arises as a representation of some known continuously differentiable function.

When $c = 0$, we refer to the power series as a Maclaurin series. In such cases, the power series takes the simpler form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Maclaurin series are useful in other areas of mathematics and engineering such as in combinatorics and probability under the section on generating functions; in electrical engineering under the name of Z-transforms and so on. A more familiar example is given below as a geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,$$

which is valid for $|x| < 1$. The exponential function ex is one of the most important examples of a function whose power series representation is often required in studies in biology, physics and other sciences:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and the sine formula

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

valid for all real x.

Negative powers are not permitted in a power series, for instance $1 + x^{-1} + x^{-2} + \cdots$ is not considered a power series (although it is a Laurent series). Similarly, fractional powers such as $x^{1/2}$ are not permitted and treated as a separate Puiseux series.

Puiseux series are a generalization of power series, first introduced by Isaac Newton in 1676 and rediscovered by Victor Puiseux in 1850, that allows for negative and fractional exponents of the indeterminate T. A **Puiseux series** in the indeterminate T is a Laurent series in $T1/n$, wheren is a positive integer. A Puiseux series may be written as:

$$\sum_{i=k}^{\infty} a_i T^{i/n},$$

where k is an integer and n is a positive integer. We state a less known theorem in relation to Puiseux series:

Theorem (Newton-Puiseux): Let P(x,y) =0 be a polynomial equation. Suppose that y = f(x), then y admits a Puiseux series expansion around a neighborhood of the origin.

In other words, every branch of an algebraic curve may be locally (in terms of x) described by a Puiseux series. The set of Puiseux series over an algebraically closed field of characteristic 0 is itself an algebraically closed field, called the **field of Puiseux series**. It is the algebraic closure of the field of Laurent series. This statement is also referred to as **Puiseux's theorem**, being an expression of the original Puiseux theorem in modern abstract language.

Approximations by Mclaurin's series provide a sense of global estimate of f(x) by smooth curves. It is possible to provide a local approximation of f(x) by fractional derivatives (which is related to the Puiseux series described above). To motivate, the proposed local approximation, we recall that $f^{(q_n)}(x) \to f(x)$ as $q_n \to 0$. We choose the sequence of rationals to be $q_n = \frac{1}{2^n}$, n=1,2,...

Since $q_n \to 0$, then given $\epsilon_1 > 0$, we can find an N>0 such that:

$$(11)\ldots \left| f^{(q_n)}(x) \to f(x) \right| < \epsilon_2$$

whenever $\left| q_n - q_{N_0} \right| < \delta$.

Illustrative Comparison of Global versus Local Approximations

In the case of the exponential function, e^x

We note that the Mclaurin's series expansion of the **exponential function** is given by:

$$f(x) = e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Applying the proposed local approximation through fractal derivatives, we have

$$f^{(q_n)}(x) = f^{(q_n)}(e^x) = f^{(q_n)}(1) + f^{(q_n)}(x) + \left(\frac{f^{(q_n)}}{2!}x^2\right) + \left(\frac{f^{(q_n)}}{3!}x^3\right) + \ldots$$

$$f^{(q_n)}(x) = 0 + \frac{1}{1!}\frac{\Gamma(2)}{\Gamma(2-q_n)}x^{1-q_n} + \frac{1}{2!}\frac{\Gamma(3)}{\Gamma(3-q_n)}x^{2-q_n} + \frac{1}{3!}\frac{\Gamma(4)}{\Gamma(4-q_n)}x^{3-q_n} + \frac{1}{4!}\frac{\Gamma(5)}{\Gamma(5-q_n)}x^{4-q_n} + \ldots$$

$$\boxed{f^{(q_n)}(x) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(m+1-q_n)} x^{m-q_n}, \qquad q_n \approx 0}$$

We illustrate this formula when the algorithm is implemented in mathematical software.

Algorithm:

1. Let f(x)= e^x and let x=1.
2. Fix error at $\varepsilon = 0.01$, we need to find q_n $in\ Q^+$. Consequently,

$$\left| f^{(q_n)}(1) \to f(1) \right| < 0.01$$

where $q_n = \frac{1}{2^n}$, $n = 1,2,3, \ldots$

and we get

$$f^{\left(\frac{1}{2^n}\right)}(x) = \sum_{m=0}^{\infty} \frac{f^{\left(\frac{1}{2^n}\right)}(x^m)}{m!}$$

$$f^{\left(\frac{1}{2^n}\right)}(1) = \sum_{m=0}^{\infty} \frac{\left(x^{m-\frac{1}{2^n}}\right)}{\Gamma\left(m+1-\frac{1}{2^n}\right)}$$

3. Here, m is defined to be the number of terms in the series to approximate the original function.

Table 1. Summary of Approximations: Global vs. Local

Function	Original value (in 5 decimal places)	Mclaurin's approximation	Local estimation through fractional derivatives ($f^{(q_n)}(x)$)	Error in Mclaurins	*Error in* Local estimation
e^1	2.71828	2.70873 *(m=5)*	2.71680 *(m=4), (n=6)*	0.00995	0.000108
e^2	7.38905	7.38095 *(m=7)*	7.38610 *(m=6), (n=3)*	0.00810	0.002950
e^3	20.0855	20.0841 *(m=11)*	20.0851 *(m=9), (n=3)*	0.00140	0.000400

We note that for the same number of terms (or even less), the values obtained using fractional derivatives are closer to the true (theoretical values).

4.0 Conclusion

The seminal idea of using the fractional series (**Puiseux**, 1850) for approximation purposes initiated more than 163 years ago was not anchored on the idea of a fractional derivative. Local approximations to functional values by fractional derivatives provide finer and better estimate than the global approximations represented by power series e.g Mclaurin's series. Fractional derivatives incorporate information on the fluctuations and irregularities near the true functional values, hence, attaining greater precision. Fractional derivative representation of a function is related to the Puiseux-Newton theorem.

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