ARIMA(p,d,q) and Non-Linear Approximation Models for the Fractal Dimension of the Density of Primes Less or Equal to a Positive Integer

*Roberto N. Padua, [1]Rodel B. Azura, [2]Mark S. Borres and [3]Adriano Patac

Abstract

The study compares the performance of the Azura et al. (2013) prediction model for the fractal dimension of the density of primes less or equal to a positive integer x with the performance of an autoregressive integrated moving average model (ARIMA(p,d,q). The actual density of primes used in this study were gathered from published table of primes . Results revealed that the time series model ARIMA(p,d,q) outperforms the Azura et al. (2013) prediction model particularly for larger values of X in the range of forecast values. The time series model is more convenient to use in practice since it only involves the previous calculated values of the fractal dimensions.

Keywords: time series model, fractal density of primes, autoregressive, moving average
AMS Classification: Number theory, applied mathematics

1.0 Introduction

Azura et al. (2013) demonstrated that the density of primes less or equal to a positive integer x can be approximated by a power-law(fractal) distribution by means of simulation. They also showed that the prediction error incurred by such a multifractal fit to the density of primes is smaller than that obtained when the Prime Number Theorem approximation is used particularly when x is of magnitude less or equal to a million (small values of x). These results are to be expected since the Prime Number Theorem is an asymptotic result which applies only when x is large. The Prime Number Theorem states that:

$$1\ldots \frac{\pi(x)}{x} \to \frac{1}{\log(x)} \text{ as } x \to \infty$$

where $\pi(x)$ is the number of primes less or equal to x, while the Multifractal Fit Hypothesis (MFH) of Azura et al. (2013) states that:

$$2\ldots \frac{\pi(x)}{x} \propto \frac{1}{x^{\lambda}} \text{ for all } x \in Z^{+}.$$

Indeed, when $\pi(x)$ is known, we can compute the exact value of λ, hereinafter referred to as the fractal dimension of x, as:

$$3\ldots \lambda = 1 - \frac{\log(\pi(x)}{\log(x)}.$$

Currently, the value of $\pi(x)$ is known up to $x = 10^{25}$ and published in various sources. It is when x exceeds this number that the approximation to the density of primes becomes of primary importance. Most algorithms depend on an unproved Riemann Hypothesis (Dudley, 2003) or on the asymptotic approximation provided by the Prime Number Theorem. In Azura et al. (2013), the known values of $\lambda(x)$ are regressed to a non-linear function of x

*Fractal Statistics Expert
[1]Mindanao University of Science and Technology
[2]University of San Jose Recoletos
[3]Surigao State College of Science and Technology

to obtain a prediction formula:

4.....log λ(x) = a + b log(x) + c (log(x))2 , x > 106 .

In their paper, they showed that the prediction error for x = 20,000 is less than 1%. The present study provides an alternative to the Azura et al. (2013) proposal by employing a time series autoregressive integrated moving average model (ARIMA(p,d,q)) using a Box-Jenkins approach (G. Box, Time Series Analysis, Forecasting and Control, 1980). Time series approaches are useful in the sense that the prediction formulae obtained are dependent only on previously computed values of the fractal dimensions.

2.0 Fractal Formalisms

In this section, we provide a brief overview of the fractal statistics formalisms introduced by Padua et al. (2012) and used in the paper of Azura et al. (2013). Let X be a random variable whose probability density function obeys the power law:

5. $$f(x) = \left(\frac{\lambda - 1}{\theta}\right)\left(\frac{x}{\theta}\right)^{-\lambda}, x \geq \theta, \theta \geq 0, \lambda > 0$$

The random variable X is then called a fractal random variable and $f(x)$ is its fractal probability distribution. The first moment of X (its mean) will not exist for $\lambda < 2$. Consequently, the second moment (its variance) will also not exist for $\lambda < 2$. The parameter λ of (6) is called the fractal dimension of X.

For $\lambda \leq 2$, the non-existence of the second moment or variance of X implies that observation from fractal distribution are highly erratic, fluctuating and rough. In fact, the Central Limit Theorem fails to apply in cases where the observation come from fractal distribution.

For $\lambda > 2$, the variance σ^2 exist and is related to λ by:

6. $\lambda = 1 + \theta\sigma$ (Padua et al. (2012))

In other words, when the variance exists, the fractal dimension λ describes the variability of the data around the mean just as the standard deviation (σ) does. Further, the fractal dimension, λ, of X is a more general description of data variability than σ.

From (6), the maximum likelihood estimator of λ is easily obtained as

7. $$\hat{\lambda} = 1 + n\left(\sum_{i=1}^{n} \log\left(\frac{x_i}{\theta}\right)\right)^{-1},$$

for $x_1, x_2, \ldots, x_n$, iid $f(x)$, Similarly, the cumulative distribution function (cdf), $F(x)$, is:

8. $$F(x) = P(X \leq x) = 1 - \left(\frac{x}{\theta}\right)^{1-\lambda}$$

Equation (8) gives the probability that an observation X is less or equal to x.

Multifractal Formalisms

The fit provided by (8) assume that there is a single exponent (fractal dimension) λ that would explain the global behaviour of $\frac{\pi(x)}{x}$. In the event that (5) proves to be large for the FF approximation using only one $\hat{\lambda}$, we modify (8) and assume several fractal dimensions (or multi fractal system). In this case, we assume that:

9. $$\frac{\pi(x)}{x} \propto \frac{1}{x^{\lambda}} \qquad \theta = 2$$

We solve for the value of λ as follows:

10. $$\lambda = 1 - \frac{\ln(\pi(x))}{\ln(x)}$$

and then obtain several approximation $\hat{F}_n(x)$:

11. $$\hat{F}_n(x) = \frac{1}{x^{\lambda}}, \quad x = 1,2,\ldots,n, n = 10^6, \lambda = \lambda(x).$$

3.0 Time Series Forecasting Models

A time series is a stochastic process {λ(t)} that depends on time t ε T. When T is discrete, we say we have a discrete time series, otherwise, the time series is continuous. The values of λ obtained by the multifractal formalisms above can be considered as realizations of a discrete time series. The series is said to be second order stationary when cov(λ(t), λ(t+k)) < ∞ for all k. In a separate paper, Padua(2012) proved that the distribution of λ(t), t= 1,2,3,... is approximately exponential and hence, the series is ipso facto second-order stationary.

For stationary time series, two popular models are the Autoregressive (AR(p)) model and the Moving Average (MA(q)) model. The p^{th} order autoregressive process assumes that the current observation is dependent on the immediate past p observations:

12. $\lambda(t) = \varphi 1\lambda(t-1) + \varphi 2\lambda(t-2) + \varphi 3\lambda(t-3) + ... + \varphi p\lambda(t-p) + \varepsilon(t), t= 2,3,...,n$

ε(t) are iid with E(ε(t)) = 0, var (ε(t)) = σ2 for all t.

Thus, an AR(1) model simply states that the current observation is a multiple of the immediate past observation: λ(t) = φ1λ(t-1) λ(t) = φ1λ(t-1) + ε(t). Equation (12) can also be used as a forecast model when treated as multiple regression (on itself) without an intercept term. Methods for estimating the weight parameters {φk} can be found in standard textbooks on time series analysis.

On the other hand, the moving average model of order q states that the current observation is a summation of weighted shocks in the qth past:

13. $\lambda(t) = \theta 1\varepsilon(t-1) + \theta 2\varepsilon(t-2) + \theta 3\varepsilon(t-3) + ... + \theta p\varepsilon(t-p) + \varepsilon(t)$

The weight parameters {θk} can likewise be computed from the data. Unlike the autoregressive model, however, (13) cannot be immediately used as a forecast function since it involves estimation of past errors. However, if we note the equivalence of (12) and (13), we can theoretically express an MA(q) model as an infinite (high order) autoregressive process and vice versa under certain conditions. These conditions are called the **invertibility conditions** discussed in time series courses.

When the original time series is not stationary, it may be possible to convert it into a stationary series through the process of **differencing**. Define the backward shift operator as:

14. $B(\lambda(t)) = \lambda(t-1),$

then the first order difference is given by:

15. $\delta(\lambda(t)) = (1-B)(\lambda(t)) = \lambda(t) - \lambda(t-1).$

Higher order differenced series can be defined recursively as follows:

16. $\delta k(\lambda(t)) = \delta k\text{-}1[\delta(\lambda(t))].$

The new series (16) is then called an integrated series. In many instances, when series are integrated, the new differenced series will become stationary.

Autocorrelation Function

An analytic way to check if the series is stationary is to view its autocorrelation function (ACF). The autocorrelation function is defined as:

17. $\rho_k = \frac{cov(X(t),X(t+k)}{sd(X(t)sd(X(t+k)}$, k = 0,1,2,...

A stationary series will exhibit a decaying autocorrelation function while a non-stationary series will display a non-decaying behaviour.

Autoregressive Integrated Moving Average Model (ARIMA(p,d,q)).

A general formulation that provides flexibility in the formulation of a time series model is to combine the AR model with the MA model on a differenced series. This model is called an ARIMA(p,d,q) which consists of a pth order autogressive model plus a qth order moving average model on a differenced series of order d. When d= 0, q =0, we have a pure AR(p) model; when d=0, p =0, we have a pure moving average model. Other combinations are now possible.

3.0 Study Design

Using the same set of primes as Azura et al. (2013), we fitted two kinds of forecast functions:

Type I (Azura et al. (2013)): logλ(t) = a + b log(X(t)) + c (log(X(t))2, and

Type II. ARIMA (p,d,q)) where p, d and q are obtained after examination of the resulting autocorrelation functions.

We subdivided the available data on the primes less than 20,000 into five (5) subsets of data:

Data 1: The primes less or equal to 4,000
Data 2: The primes less or equal to 8,000
Data 3: The primes less or equal to 12,000
Data 4: The primes less or equal to 16,000
Data 5: The primes less or equal to 20,000

For each data set, we computed the Type I and Type II estimates of the fractal dimensions. The estimates of the fractal dimensions form the time series of observations {λ(t)}.

Since the number of primes less or equal to 23,000 are available, we forecasted the

Forecast 1: Values of X from 4001 to 4020
Forecast 2: Values of X from 8001 to 8020
Forecast 3: Values of X from 12001 to 12020
Forecast 4: Values of X from 20001 to 20020

using Type I and Type II forecast functions.

The mean absolute prediction errors (MAPE) were computed for each of the different forecast sets above. The basis for comparison is the absolute deviation from the actual density of primes less or equal to x which is available.

4.0 Results and Discussion

4.1 Data Set 1: X = 2 to X = 4000, Base data: log(λ(t))

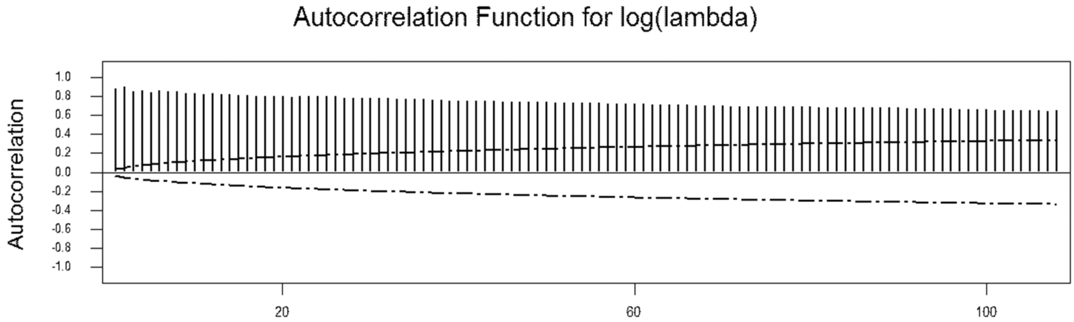
Data for the density of primes less or equal to X, 2 < X < 4,000 were used to generate the Azura forecast function. The forecast function obtained was:

log(lambda) = - 0.946 - 0.0589 lnX

S = 0.01826 R-Sq = 91.0% R-Sq(adj) = 91.0%

This forecast function was subsequently used to generate the forecasted values of log(lambda) from 4001 to 4020.

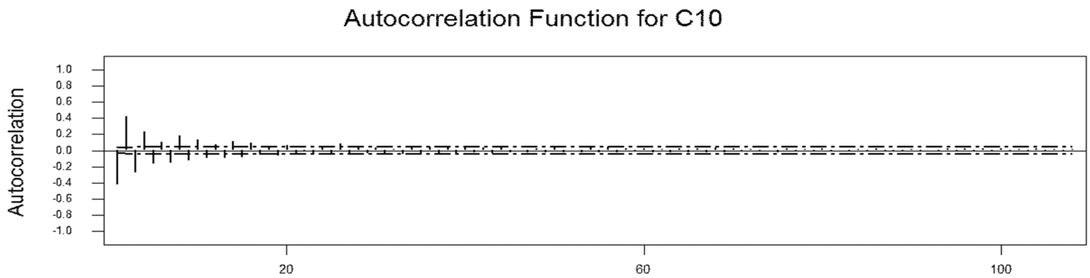
The autocorrelation function for the values of log(lambda) revealed a non-stationary series. This signals the use of differencing. The graph of the autocorrelation function is given Figure 1.



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.88	55.75	3110.80	16	0.81	10.83	4.6E+04	31	0.78	7.66	8.4E+04	46	0.74	6.13	1.2E+05	61	0.72	5.24	1.5E+05
2	0.90	35.68	6365.84	17	0.80	10.39	4.8E+04	32	0.78	7.55	8.6E+04	47	0.74	6.07	1.2E+05	62	0.72	5.20	1.5E+05
3	0.85	26.22	9243.91	18	0.81	10.20	5.1E+04	33	0.78	7.41	8.8E+04	48	0.74	6.02	1.2E+05	63	0.71	5.15	1.6E+05
4	0.87	23.09	1.2E+04	19	0.80	9.91	5.3E+04	34	0.78	7.30	9.1E+04	49	0.74	5.94	1.3E+05	64	0.71	5.11	1.6E+05
5	0.84	19.89	1.5E+04	20	0.80	9.68	5.6E+04	35	0.77	7.14	9.3E+04	50	0.74	5.88	1.3E+05	65	0.71	5.04	1.6E+05
6	0.86	18.59	1.8E+04	21	0.79	9.35	5.8E+04	36	0.77	7.06	9.6E+04	51	0.73	5.79	1.3E+05	66	0.71	5.02	1.6E+05
7	0.85	17.05	2.1E+04	22	0.80	9.19	6.1E+04	37	0.76	6.94	9.8E+04	52	0.73	5.74	1.3E+05	67	0.71	4.96	1.6E+05
8	0.85	15.96	2.4E+04	23	0.80	9.01	6.4E+04	38	0.76	6.84	1.0E+05	53	0.73	5.69	1.3E+05	68	0.71	4.93	1.7E+05
9	0.83	14.58	2.7E+04	24	0.80	8.88	6.6E+04	39	0.76	6.69	1.0E+05	54	0.73	5.65	1.4E+05	69	0.70	4.86	1.7E+05
10	0.83	13.96	2.9E+04	25	0.80	8.66	6.9E+04	40	0.76	6.63	1.1E+05	55	0.73	5.58	1.4E+05	70	0.70	4.83	1.7E+05
11	0.82	13.08	3.2E+04	26	0.80	8.49	7.1E+04	41	0.75	6.50	1.1E+05	56	0.73	5.53	1.4E+05	71	0.70	4.77	1.7E+05
12	0.83	12.69	3.5E+04	27	0.79	8.22	7.4E+04	42	0.75	6.45	1.1E+05	57	0.72	5.43	1.4E+05	72	0.70	4.75	1.7E+05
13	0.82	12.13	3.8E+04	28	0.79	8.09	7.6E+04	43	0.75	6.36	1.1E+05	58	0.72	5.39	1.4E+05	73	0.69	4.71	1.8E+05
14	0.82	11.72	4.0E+04	29	0.78	7.89	7.9E+04	44	0.75	6.29	1.1E+05	59	0.72	5.32	1.5E+05	74	0.69	4.69	1.8E+05
15	0.81	11.11	4.3E+04	30	0.78	7.81	8.1E+04	45	0.74	6.18	1.2E+05	60	0.72	5.29	1.5E+05	75	0.69	4.65	1.8E+05

Figure 1: Autocorrelation for Raw Data

The graph of the differenced series, however, showed that the TACF dies out rapidly. It follows that the first order differenced series is a stationary time series which allows for the fitting of a time series forecast model.



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	-0.42	-26.79	718.10	16	0.09	3.89	2688.62	31	-0.04	-1.47	2851.01	46	0.00	0.18	2932.38	61	-0.02	-0.81	2977.03
2	0.42	22.77	1423.41	17	-0.06	-2.43	2702.49	32	0.01	0.49	2851.59	47	-0.02	-1.00	2934.86	62	0.01	0.41	2977.45
3	-0.27	-13.09	1716.81	18	0.05	1.95	2711.47	33	-0.04	-1.79	2859.41	48	0.03	1.13	2938.06	63	-0.03	-1.00	2980.00
4	0.24	10.98	1941.01	19	-0.06	-2.62	2727.69	34	0.05	2.14	2870.65	49	-0.03	-1.15	2941.34	64	0.03	1.09	2983.01
5	-0.17	-7.58	2054.44	20	0.07	2.81	2746.51	35	-0.04	-1.43	2875.70	50	0.04	1.43	2946.47	65	-0.02	-0.83	2984.75
6	0.11	4.80	2101.16	21	-0.04	-1.84	2754.58	36	0.04	1.75	2883.20	51	-0.02	-0.85	2948.29	66	0.03	1.16	2988.15
7	-0.16	-6.91	2199.14	22	-0.00	-0.12	2754.62	37	-0.04	-1.69	2890.21	52	0.00	0.05	2948.29	67	-0.02	-0.98	2990.60
8	0.18	8.02	2334.63	23	-0.05	-2.04	2764.57	38	0.05	1.97	2899.79	53	-0.03	-1.01	2950.84	68	0.03	1.38	2995.46
9	-0.13	-5.41	2398.14	24	0.06	2.34	2777.71	39	-0.04	-1.48	2905.18	54	0.03	1.25	2954.72	69	-0.02	-0.82	2997.16
10	0.14	5.85	2473.58	25	-0.06	-2.36	2791.08	40	0.05	2.07	2915.81	55	-0.03	-1.20	2958.32	70	0.03	1.03	2999.86
11	-0.09	-3.88	2507.34	26	0.08	3.32	2817.60	41	-0.03	-1.09	2918.74	56	0.05	1.86	2967.03	71	-0.02	-0.60	3000.79
12	0.08	3.26	2531.29	27	-0.05	-1.92	2826.51	42	0.03	1.26	2922.70	57	-0.02	-0.99	2969.51	72	0.01	0.57	3001.61
13	-0.10	-4.16	2570.66	28	0.05	1.90	2835.27	43	-0.03	-1.24	2926.51	58	0.02	0.88	2971.47	73	-0.02	-0.66	3002.70
14	0.12	4.92	2626.07	29	-0.03	-1.40	2840.01	44	0.03	1.22	2930.24	59	-0.02	-0.75	2972.91	74	0.00	0.18	3002.78
15	-0.08	-3.43	2653.32	30	0.04	1.54	2845.77	45	-0.02	-0.91	2932.30	60	0.02	0.99	2975.36	75	-0.02	-0.82	3004.51

Figure 2: Autocorrelation Function for First Order Differenced Series

The first order differenced series was modelled as an autoregressive process of order 1 (ARIMA(1,1,0). Trials over higher order AR processes and MA process revealed no significant improvements in the predictive ability of the AR(1,1,0) model. The Azura forecasts are compared with the ARIMA(1,1,0) forecasts in table 1.

Table 1: Forecast Values for ARIMA (1,1,0), Azura Model and Actual Values of the Density

Forecast Origin	ARIMA (1,1,0)	Azura Forecast	Actual Density	ARIMA Error	AZURA Error
4000 to 4019	-1.43039	-1.43452	-1.43037	0.0000164	0.0041495
	-1.43115	-1.43453	-1.43117	0.0000161	0.0033642
	-1.43040	-1.43455	-1.43108	0.0006779	0.0034690
	-1.43114	-1.43456	-1.43192	0.0007815	0.0026437
	-1.43042	-1.43458	-1.43179	0.0013728	0.0027884
	-1.43112	-1.43459	-1.43171	0.0005863	0.0028831
	-1.43043	-1.43461	-1.43163	0.0011983	0.0029778
	-1.43111	-1.43462	-1.43242	0.0013105	0.0022025
	-1.43045	-1.43464	-1.43234	0.0018944	0.0022972
	-1.43110	-1.43465	-1.43225	0.0011541	0.0024019
	-1.43046	-1.43467	-1.43213	0.0016710	0.0025366
	-1.43108	-1.43468	-1.43205	0.0009672	0.0026313
	-1.43047	-1.43470	-1.43196	0.0014882	0.0027360
	-1.43107	-1.43471	-1.43276	0.0016897	0.0019506
	-1.43048	-1.43473	-1.43267	0.0021860	0.0020553
	-1.43106	-1.43474	-1.43259	0.0015317	0.0021500
	-1.43050	-1.43475	-1.43246	0.0019642	0.0022947
	-1.43105	-1.43477	-1.43238	0.0013332	0.0023893
	-1.43051	-1.43478	-1.43230	0.0017929	0.0024840
	-1.43104	-1.43480	-1.43309	0.0020543	0.0017086
			MEAN ABSOLUTE PREDICTION ERROR:	0.00128	0.00261
			STANDARD ERROR OF THE MEAN:	0.00014	0.00013

Comparison of the mean absolute prediction errors revealed that the ARIMA(1,1,0) outperformed the Azura model by over 200%. An examination of the forecast errors revealed the pattern of movements of the fractal dimensions of the actual density of primes is synchronized with the movements of the ARIMA forecasts while the Azura forecasts formed a smooth function way below the actual movements of the actual density fractal dimensions.

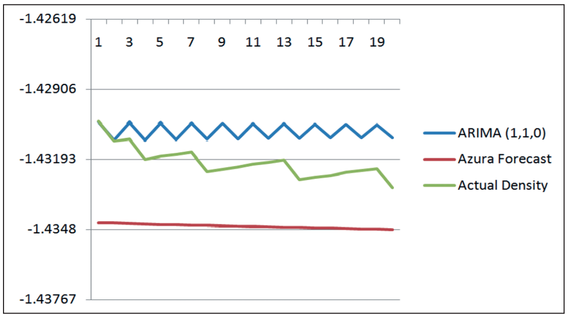


Figure 3: Forecast Values for ARIMA(1,1,0), AZURA Forecasts and Actual Density

Data Set 2: X = 2 to X = 8000, Base data: log(λ(t))

The Azura forecast function was similarly computed for 2 < X <8,000 and is provided below:

log(lambda) = - 0.960 - 0.0568 lnX

S = 0.01310 R-Sq = 94.9% R-Sq(adj) = 94.9%

The graph of the autocorrelation function for log(lambda) is displayed below: The Azura forecast function was similarly computed for 2 < X <8,000 and is provided below:

log(lambda) = - 0.960 - 0.0568 lnX

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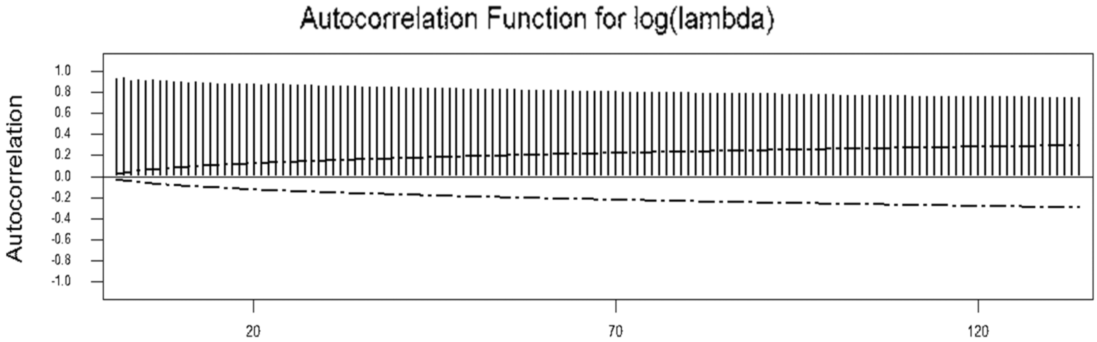
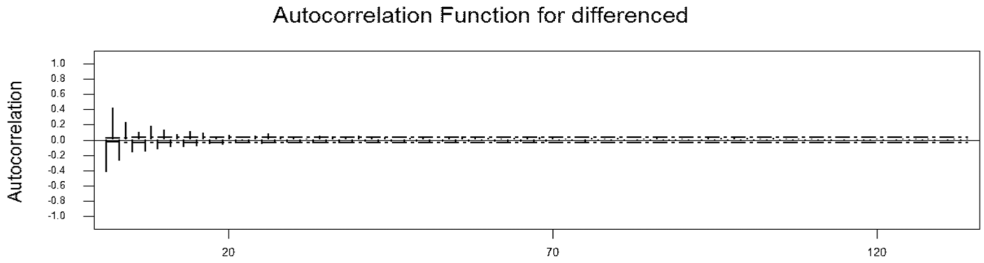


Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.93	83.39	6956.62	16	0.89	15.62	1.1E+05	31	0.86	11.06	2.0E+05	46	0.84	8.93	2.8E+05	61	0.82	7.68	3.7E+05
2	0.94	50.96	1.4E+04	17	0.88	15.06	1.1E+05	32	0.86	10.89	2.0E+05	47	0.84	8.84	2.9E+05	62	0.82	7.61	3.7E+05
3	0.91	38.38	2.1E+04	18	0.88	14.70	1.2E+05	33	0.86	10.70	2.1E+05	48	0.84	8.76	3.0E+05	63	0.82	7.54	3.8E+05
4	0.92	33.15	2.8E+04	19	0.88	14.29	1.2E+05	34	0.86	10.55	2.2E+05	49	0.84	8.65	3.0E+05	64	0.82	7.48	3.8E+05
5	0.91	28.88	3.4E+04	20	0.88	13.94	1.3E+05	35	0.86	10.35	2.2E+05	50	0.84	8.57	3.1E+05	65	0.81	7.41	3.9E+05
6	0.92	26.57	4.1E+04	21	0.88	13.52	1.4E+05	36	0.86	10.22	2.3E+05	51	0.83	8.45	3.1E+05	66	0.81	7.36	3.9E+05
7	0.91	24.40	4.8E+04	22	0.88	13.25	1.4E+05	37	0.85	10.05	2.3E+05	52	0.83	8.38	3.2E+05	67	0.81	7.29	4.0E+05
8	0.91	22.77	5.4E+04	23	0.88	12.97	1.5E+05	38	0.85	9.92	2.4E+05	53	0.83	8.29	3.2E+05	68	0.81	7.24	4.0E+05
9	0.90	21.08	6.1E+04	24	0.88	12.73	1.5E+05	39	0.85	9.74	2.4E+05	54	0.83	8.23	3.3E+05	69	0.81	7.16	4.1E+05
10	0.90	20.06	6.7E+04	25	0.88	12.44	1.6E+05	40	0.85	9.63	2.5E+05	55	0.83	8.14	3.3E+05	70	0.81	7.11	4.2E+05
11	0.89	18.93	7.4E+04	26	0.87	12.19	1.7E+05	41	0.84	9.48	2.6E+05	56	0.83	8.06	3.4E+05	71	0.80	7.05	4.2E+05
12	0.90	18.23	8.0E+04	27	0.87	11.88	1.7E+05	42	0.84	9.38	2.6E+05	57	0.82	7.95	3.5E+05	72	0.80	7.00	4.3E+05
13	0.89	17.45	8.6E+04	28	0.87	11.67	1.8E+05	43	0.84	9.26	2.7E+05	58	0.82	7.89	3.5E+05	73	0.80	6.95	4.3E+05
14	0.89	16.82	9.3E+04	29	0.86	11.42	1.9E+05	44	0.84	9.15	2.7E+05	59	0.82	7.80	3.6E+05	74	0.80	6.91	4.4E+05
15	0.88	16.08	9.9E+04	30	0.87	11.26	1.9E+05	45	0.84	9.02	2.8E+05	60	0.82	7.75	3.6E+05	75	0.80	6.85	4.4E+05

Figure 4: Autocorrelation function for raw data

A causal perusal of the autocorrelation function again showed high degree of non-stationarity for which reason we took the first order differenced series and plotted the autocorrelation function of the differened series. The graph is shown below:



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	-0.42	-37.87	1434.91	16	0.09	5.50	5371.99	31	-0.04	-2.07	5695.78	46	0.00	0.25	5857.77	61	-0.02	-1.15	5946.51
2	0.42	32.21	2844.82	17	-0.06	-3.43	5399.65	32	0.01	0.69	5696.95	47	-0.02	-1.41	5862.69	62	0.01	0.58	5947.35
3	-0.27	-18.50	3430.97	18	0.05	2.76	5417.58	33	-0.04	-2.53	5712.50	48	0.03	1.61	5869.08	63	-0.03	-1.42	5952.40
4	0.24	15.53	3879.03	19	-0.06	-3.70	5449.91	34	0.05	3.03	5734.91	49	-0.03	-1.62	5875.59	64	0.03	1.54	5958.38
5	-0.17	-10.72	4105.61	20	0.07	3.98	5487.47	35	-0.04	-2.03	5744.93	50	0.04	2.03	5885.80	65	-0.02	-1.17	5961.82
6	0.11	6.79	4199.00	21	-0.04	-2.60	5503.57	36	0.04	2.47	5759.90	51	-0.02	-1.20	5889.40	66	0.03	1.64	5968.58
7	-0.16	-9.76	4394.61	22	-0.00	-0.17	5503.64	37	-0.04	-2.38	5773.84	52	0.00	0.08	5889.41	67	-0.02	-1.39	5973.45
8	0.18	11.35	4665.30	23	-0.05	-2.88	5523.47	38	0.05	2.79	5792.91	53	-0.03	-1.42	5894.46	68	0.03	1.96	5983.08
9	-0.13	-7.65	4792.06	24	0.06	3.31	5549.69	39	-0.04	-2.09	5803.64	54	0.03	1.76	5902.20	69	-0.02	-1.16	5986.46
10	0.14	8.28	4942.76	25	-0.06	-3.33	5576.31	40	0.05	2.93	5824.81	55	-0.03	-1.69	5909.33	70	0.03	1.46	5991.82
11	-0.09	-5.49	5010.13	26	0.08	4.69	5629.20	41	-0.03	-1.54	5830.62	56	0.05	2.64	5926.65	71	-0.02	-0.85	5993.66
12	0.08	4.61	5058.00	27	-0.05	-2.71	5646.95	42	0.03	1.79	5838.53	57	-0.02	-1.40	5931.56	72	0.01	0.80	5995.29
13	-0.10	-5.88	5136.53	28	0.05	2.69	5664.41	43	-0.03	-1.75	5846.10	58	0.02	1.25	5935.47	73	-0.02	-0.93	5997.45
14	0.12	6.95	5247.16	29	-0.03	-1.97	5673.85	44	0.03	1.73	5853.53	59	-0.02	-1.07	5938.32	74	0.00	0.25	5997.61
15	-0.08	-4.84	5301.52	30	0.04	2.18	5685.35	45	-0.02	-1.28	5857.61	60	0.02	1.40	5943.20	75	-0.02	-1.16	6001.02

Figure 5: Autocorrelation Function for First Order Differenced Series

The autocorrelation function of the first order differenced series displayed a rapidly decaying autocorrelations. This means that the series is now stationary allowing for a time series model fit. We tried out possible values of p,d, and q in ARIMA(p,d,q) and found that the choices p = 1, d = 1, q = 0 remained the best possible choices. Thus, an ARIMA(1,1,0) was fitted on the data and forecast values for X = 8,001 to X = 8,020 were computed. The results are displayed below:

Table 2: Forecast Values for ARIMA, Azura model and Actual Density

Xnew	azura forecasts	ARIMA Forecasts	Density actual	Azura Error	ARIMA Error
8001	-1.47048	-1.43039	-1.46707	0.0034099	0.0366835
8002	-1.47049	-1.43115	-1.46703	0.003457	0.0358762
8003	-1.47049	-1.4304	-1.46698	0.0035141	0.0365776
8004	-1.4705	-1.43114	-1.46694	0.0035612	0.0358017
8005	-1.47051	-1.43042	-1.4669	0.0036083	0.0364825
8006	-1.47052	-1.43112	-1.46681	0.0037054	0.0356866
8007	-1.47052	-1.43043	-1.46677	0.0037525	0.0363379
8008	-1.47053	-1.43111	-1.46672	0.0038096	0.0356108
8009	-1.47054	-1.43045	-1.46668	0.0038566	0.0362339
8010	-1.47054	-1.4311	-1.46711	0.0034337	0.0360145
8011	-1.47055	-1.43046	-1.46707	0.0034808	0.0366105
8012	-1.47056	-1.43108	-1.4675	0.0030579	0.0364176
8013	-1.47057	-1.43047	-1.46746	0.003105	0.0369877
8014	-1.47057	-1.43107	-1.46742	0.0031521	0.0363502
8015	-1.47058	-1.43048	-1.46737	0.0032092	0.0368853
8016	-1.47059	-1.43106	-1.46733	0.0032563	0.0362723
8017	-1.47059	-1.4305	-1.46729	0.0033034	0.0367935
8018	-1.4706	-1.43105	-1.46772	0.0028804	0.0366739
8019	-1.47061	-1.43051	-1.46768	0.0029275	0.0371721
8020	-1.47061	-1.43103	-1.46763	0.0029846	0.036595
	MEAN PREDICTION ERROR:			0.00337	0.03640
	STANDARD ERROR OF THE MEAN:			0.00007	0.00010

For this sample size, it appears that the Azura model outperforms the ARIMA(1,1,0) model by over 100% in terms of forecast accuracy. A graph of the forecasts is shown below:

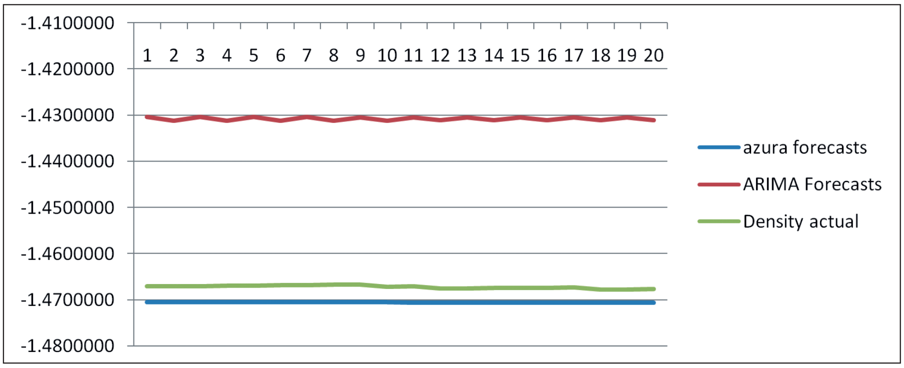


Figure 4: ACTUAL DENSITY, ARIMA and AZURA FORECASTS

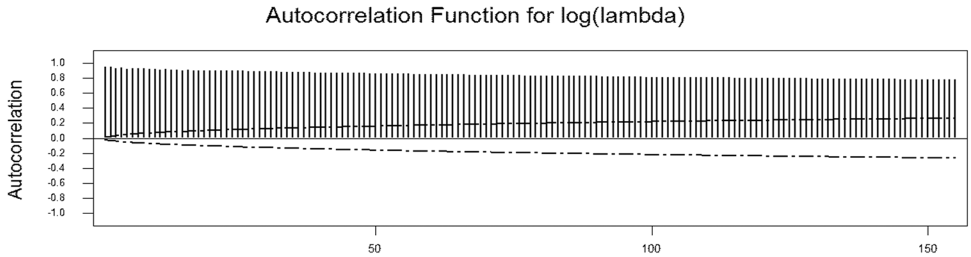
Data Set 3: X =2 to X = 12000 base data: log(λ(t))

The Azura forecast function is provided below:

log(lambda) = - 0.736 - 0.127 lnX + 0.00517 lnX-square

S = 0.01695 R-Sq = 89.9% R-Sq(adj) = 89.8%

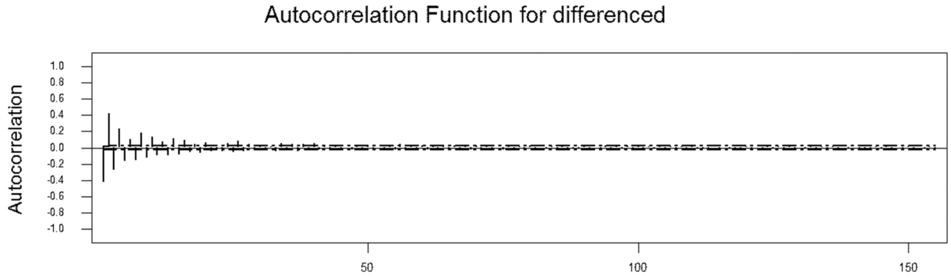
while the autocorrelation function of the raw data is displayed below:



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.95	103.63	1.1E+04	16	0.91	19.23	1.6E+05	31	0.89	13.64	3.1E+05	46	0.87	11.04	4.5E+05	61	0.85	9.50	5.8E+05
2	0.95	62.58	2.2E+04	17	0.90	18.56	1.7E+05	32	0.89	13.42	3.2E+05	47	0.87	10.92	4.6E+05	62	0.85	9.43	5.9E+05
3	0.93	47.42	3.2E+04	18	0.90	18.10	1.8E+05	33	0.89	13.20	3.3E+05	48	0.87	10.82	4.7E+05	63	0.85	9.34	6.0E+05
4	0.94	40.76	4.3E+04	19	0.90	17.59	1.9E+05	34	0.89	13.00	3.4E+05	49	0.87	10.70	4.8E+05	64	0.85	9.27	6.1E+05
5	0.92	35.60	5.3E+04	20	0.90	17.16	2.0E+05	35	0.88	12.77	3.5E+05	50	0.86	10.59	4.8E+05	65	0.85	9.18	6.2E+05
6	0.93	32.64	6.3E+04	21	0.90	16.66	2.1E+05	36	0.88	12.60	3.6E+05	51	0.86	10.45	4.9E+05	66	0.85	9.12	6.3E+05
7	0.93	29.98	7.4E+04	22	0.90	16.31	2.2E+05	37	0.88	12.41	3.7E+05	52	0.86	10.36	5.0E+05	67	0.85	9.04	6.3E+05
8	0.93	27.96	8.4E+04	23	0.90	15.97	2.3E+05	38	0.88	12.24	3.8E+05	53	0.86	10.26	5.1E+05	68	0.84	8.97	6.4E+05
9	0.92	25.96	9.4E+04	24	0.90	15.66	2.4E+05	39	0.87	12.03	3.8E+05	54	0.86	10.17	5.2E+05	69	0.84	8.88	6.5E+05
10	0.92	24.67	1.0E+05	25	0.90	15.31	2.5E+05	40	0.87	11.89	3.9E+05	55	0.86	10.06	5.3E+05	70	0.84	8.82	6.6E+05
11	0.91	23.32	1.1E+05	26	0.90	15.00	2.6E+05	41	0.87	11.71	4.0E+05	56	0.86	9.97	5.4E+05	71	0.84	8.74	6.7E+05
12	0.92	22.42	1.2E+05	27	0.89	14.64	2.7E+05	42	0.87	11.59	4.1E+05	57	0.85	9.85	5.5E+05	72	0.84	8.69	6.8E+05
13	0.91	21.47	1.3E+05	28	0.89	14.38	2.8E+05	43	0.87	11.44	4.2E+05	58	0.85	9.76	5.6E+05	73	0.84	8.62	6.9E+05
14	0.91	20.69	1.4E+05	29	0.89	14.09	2.9E+05	44	0.87	11.31	4.3E+05	59	0.85	9.66	5.6E+05	74	0.84	8.57	6.9E+05
15	0.91	19.82	1.5E+05	30	0.89	13.88	3.0E+05	45	0.87	11.15	4.4E+05	60	0.85	9.59	5.7E+05	75	0.84	8.50	7.0E+05

Figure 6: Autocorrelation Function of Original Data

Again, the autocorrelation function for the original raw data, log(lambda), displayed non-stationarity with the autocorrelations displaying no indications of decaying. The autocorrelation function of the differenced series is shown below:



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	-0.42	-46.38	2151.60	16	0.09	6.74	8055.25	31	-0.04	-2.54	8540.41	46	0.00	0.31	8783.00	61	-0.02	-1.41	8915.81
2	0.42	39.45	4266.20	17	-0.06	-4.20	8096.69	32	0.01	0.85	8542.16	47	-0.02	-1.73	8790.36	62	0.01	0.71	8917.08
3	-0.27	-22.66	5145.02	18	0.05	3.38	8123.58	33	-0.04	-3.09	8565.43	48	0.03	1.97	8799.95	63	-0.03	-1.74	8924.63
4	0.24	19.02	5817.00	19	-0.06	-4.53	8172.00	34	0.05	3.71	8599.02	49	-0.03	-1.98	8809.68	64	0.03	1.89	8933.58
5	-0.17	-13.13	6156.69	20	0.07	4.88	8228.33	35	-0.04	-2.48	8614.02	50	0.04	2.48	8824.97	65	-0.02	-1.43	8938.72
6	0.11	8.31	6296.78	21	-0.04	-3.19	8252.43	36	0.04	3.03	8636.46	51	-0.02	-1.47	8830.35	66	0.03	2.01	8948.85
7	-0.16	-11.96	6589.96	22	-0.00	-0.21	8252.54	37	-0.04	-2.92	8657.31	52	0.00	0.09	8830.37	67	-0.02	-1.70	8956.12
8	0.18	13.90	6995.90	23	-0.05	-3.53	8282.22	38	0.05	3.42	8685.89	53	-0.03	-1.74	8837.91	68	0.03	2.40	8970.54
9	-0.13	-9.36	7185.87	24	0.06	4.06	8321.54	39	-0.04	-2.56	8701.94	54	0.03	2.16	8849.51	69	-0.02	-1.42	8975.58
10	0.14	10.14	7411.85	25	-0.06	-4.08	8361.40	40	0.05	3.59	8733.65	55	-0.03	-2.07	8860.17	70	0.03	1.79	8983.61
11	-0.09	-6.72	7512.81	26	0.08	5.75	8440.66	41	-0.03	-1.88	8742.35	56	0.05	3.23	8886.10	71	-0.02	-1.05	8986.36
12	0.08	5.64	7584.61	27	-0.05	-3.32	8467.24	42	0.03	2.19	8754.20	57	-0.02	-1.72	8893.44	72	0.01	0.99	8988.80
13	-0.10	-7.21	7702.27	28	0.05	3.29	8493.42	43	-0.03	-2.14	8765.53	58	0.02	1.53	8899.30	73	-0.02	-1.13	8992.02
14	0.12	8.52	7868.16	29	-0.03	-2.42	8507.56	44	0.03	2.12	8776.66	59	-0.02	-1.31	8903.56	74	0.00	0.31	8992.27
15	-0.08	-5.93	7949.59	30	0.04	2.67	8524.80	45	-0.02	-1.57	8782.76	60	0.02	1.71	8910.88	75	-0.02	-1.42	8997.36

Figure 7: Autocorrelation Function of Differenced Series

An ARIMA(1,1,0) turned out to be the best among the choices we made to model the differenced series. The forecast errors incurred using this model are provided below together with the forecast errors of the Azura function.

Table 3: Forecast Errors of ARIMA, Azura Models

X NEW	Azura For.	ARIMA FORECASTS	ACTUAL	Azura Error	ARIMA Error
12001	-1.47276	-1.41634	-1.41634	0.056422	8E-07
12002	-1.47276	-1.4163	-1.4163	0.056465	8E-07
12003	-1.47277	-1.41634	-1.41626	0.056507	7.84E-05
12004	-1.47277	-1.4163	-1.41622	0.05655	8.16E-05
12005	-1.47277	-1.41634	-1.41622	0.056552	0.000118
12006	-1.47277	-1.4163	-1.41618	0.056595	0.000122
12007	-1.47278	-1.41634	-1.41614	0.056637	0.000197
12008	-1.47278	-1.4163	-1.41614	0.05664	0.000163
12009	-1.47278	-1.41634	-1.41609	0.056692	0.000246
12010	-1.47278	-1.4163	-1.41605	0.056735	0.000254
12011	-1.47279	-1.41634	-1.41605	0.056737	0.000286
12012	-1.47279	-1.4163	-1.41601	0.05678	0.000294
12013	-1.47279	-1.41633	-1.41597	0.056822	0.000365
12014	-1.47279	-1.41631	-1.41597	0.056825	0.000335
12015	-1.4728	-1.41633	-1.41593	0.056867	0.000404
12016	-1.4728	-1.41631	-1.41589	0.05691	0.000416
12017	-1.4728	-1.41633	-1.41589	0.056912	0.000444
12018	-1.4728	-1.41631	-1.41585	0.056955	0.000456
12019	-1.47281	-1.41633	-1.41581	0.056997	0.000523
12020	-1.47281	-1.41631	-1.41581	0.057	0.000497
	MEAN ABSOLUTE PREDICTION ERROR:			0.05673	0.00026
	STANDARD ERROR OF THE MEAN:			0.00004	0.00004

The ARIMA(1,1,0) model incurred a lower mean absolute prediction error than the Azura model. In fact, its accuracy is patently more pronounced than the Azura prediction.

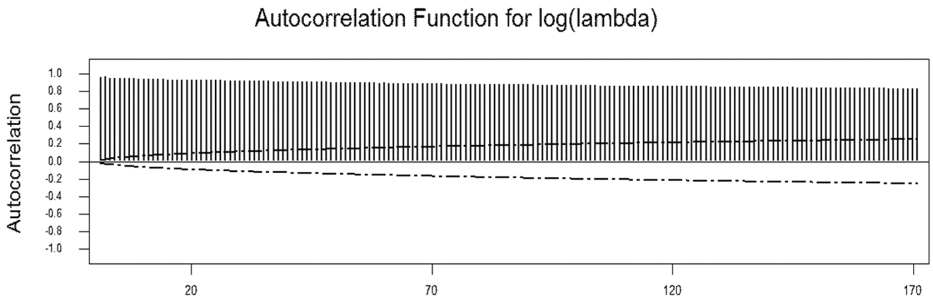
Data Set 4: X =2 to X=16000, base data: logλ(t)

The Azura forecast function is listed below:

log(lambda) = - 0.269 - 0.276 lnX + 0.0164 lnX-square

S = 0.03847 R-Sq = 50.3% R-Sq(adj) = 50.3%

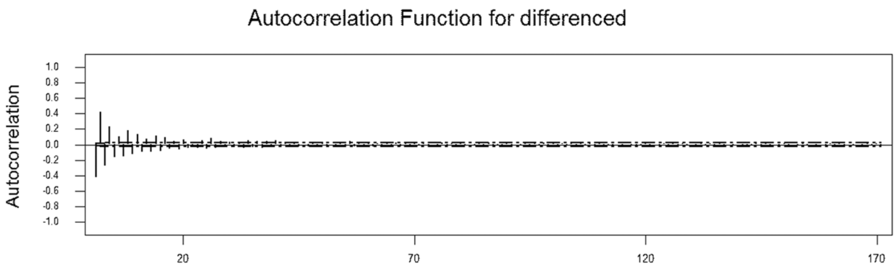
The autocorrelation function of the original raw data is displayed below:



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.96	121.66	1.5E+04	16	0.93	22.37	2.3E+05	31	0.92	15.89	4.4E+05	46	0.91	12.91	6.4E+05	61	0.90	11.14	8.3E+05
2	0.97	72.51	3.0E+04	17	0.93	21.62	2.4E+05	32	0.92	15.64	4.5E+05	47	0.91	12.77	6.5E+05	62	0.89	11.05	8.5E+05
3	0.95	55.31	4.4E+04	18	0.93	21.05	2.6E+05	33	0.92	15.38	4.6E+05	48	0.91	12.65	6.6E+05	63	0.89	10.95	8.6E+05
4	0.96	47.30	5.9E+04	19	0.93	20.47	2.7E+05	34	0.92	15.15	4.8E+05	49	0.91	12.51	6.8E+05	64	0.89	10.87	8.7E+05
5	0.95	41.44	7.3E+04	20	0.93	19.95	2.9E+05	35	0.92	14.90	4.9E+05	50	0.90	12.38	6.9E+05	65	0.89	10.77	8.8E+05
6	0.95	37.83	8.8E+04	21	0.93	19.40	3.0E+05	36	0.92	14.70	5.0E+05	51	0.90	12.23	7.0E+05	66	0.89	10.69	9.0E+05
7	0.95	34.76	1.0E+05	22	0.93	18.98	3.1E+05	37	0.92	14.48	5.2E+05	52	0.90	12.12	7.2E+05	67	0.89	10.60	9.1E+05
8	0.95	32.40	1.2E+05	23	0.93	18.57	3.3E+05	38	0.91	14.28	5.3E+05	53	0.90	12.00	7.3E+05	68	0.89	10.52	9.2E+05
9	0.94	30.20	1.3E+05	24	0.93	18.20	3.4E+05	39	0.91	14.06	5.4E+05	54	0.90	11.90	7.4E+05	69	0.89	10.43	9.3E+05
10	0.94	28.66	1.5E+05	25	0.93	17.80	3.5E+05	40	0.91	13.89	5.6E+05	55	0.90	11.78	7.6E+05	70	0.89	10.35	9.5E+05
11	0.94	27.14	1.6E+05	26	0.93	17.45	3.7E+05	41	0.91	13.69	5.7E+05	56	0.90	11.67	7.7E+05	71	0.89	10.27	9.6E+05
12	0.94	26.04	1.7E+05	27	0.92	17.05	3.8E+05	42	0.91	13.54	5.8E+05	57	0.90	11.54	7.8E+05	72	0.89	10.20	9.7E+05
13	0.94	24.95	1.9E+05	28	0.92	16.75	4.0E+05	43	0.91	13.37	6.0E+05	58	0.90	11.44	7.9E+05	73	0.89	10.13	9.9E+05
14	0.94	24.03	2.0E+05	29	0.92	16.42	4.1E+05	44	0.91	13.22	6.1E+05	59	0.90	11.33	8.1E+05	74	0.89	10.06	1.0E+06
15	0.93	23.08	2.2E+05	30	0.92	16.17	4.2E+05	45	0.91	13.05	6.2E+05	60	0.90	11.24	8.2E+05	75	0.88	9.99	1.0E+06

Figure 8: Autocorrelation Function of Raw Data

Since the original raw data displayed non-stationarity, we differenced once to obtain the autocorrelation function below:



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	-0.42	-53.55	2868.27	16	0.09	7.78	1.1E+04	31	-0.04	-2.93	1.1E+04	46	0.00	0.36	1.2E+04	61	-0.02	-1.62	1.2E+04
2	0.42	45.55	5687.65	17	-0.06	-4.85	1.1E+04	32	0.01	0.98	1.1E+04	47	-0.02	-1.99	1.2E+04	62	0.01	0.82	1.2E+04
3	-0.27	-26.16	6859.13	18	0.05	3.90	1.1E+04	33	-0.04	-3.57	1.1E+04	48	0.03	2.27	1.2E+04	63	-0.02	-2.01	1.2E+04
4	0.24	21.96	7755.06	19	-0.06	-5.23	1.1E+04	34	0.05	4.29	1.1E+04	49	-0.03	-2.29	1.2E+04	64	0.03	2.19	1.2E+04
5	-0.17	-15.16	8207.84	20	0.07	5.64	1.1E+04	35	-0.04	-2.86	1.1E+04	50	0.04	2.87	1.2E+04	65	-0.02	-1.65	1.2E+04
6	0.11	9.60	8394.62	21	-0.04	-3.68	1.1E+04	36	0.04	3.50	1.2E+04	51	-0.02	-1.70	1.2E+04	66	0.03	2.32	1.2E+04
7	-0.16	-13.81	8785.36	22	-0.00	-0.24	1.1E+04	37	-0.04	-3.37	1.2E+04	52	0.00	0.11	1.2E+04	67	-0.02	-1.97	1.2E+04
8	0.18	16.06	9326.57	23	-0.05	-4.08	1.1E+04	38	0.05	3.95	1.2E+04	53	-0.02	-2.01	1.2E+04	68	0.03	2.77	1.2E+04
9	-0.13	-10.81	9579.75	24	0.06	4.69	1.1E+04	39	-0.04	-2.95	1.2E+04	54	0.03	2.50	1.2E+04	69	-0.02	-1.64	1.2E+04
10	0.14	11.71	9881.03	25	-0.06	-4.71	1.1E+04	40	0.05	4.15	1.2E+04	55	-0.03	-2.39	1.2E+04	70	0.03	2.07	1.2E+04
11	-0.09	-7.76	1.0E+04	26	0.08	6.64	1.1E+04	41	-0.03	-2.17	1.2E+04	56	0.05	3.73	1.2E+04	71	-0.02	-1.21	1.2E+04
12	0.08	6.52	1.0E+04	27	-0.05	-3.83	1.1E+04	42	0.03	2.53	1.2E+04	57	-0.02	-1.98	1.2E+04	72	0.01	1.14	1.2E+04
13	-0.10	-8.32	1.0E+04	28	0.05	3.80	1.1E+04	43	-0.03	-2.47	1.2E+04	58	0.02	1.77	1.2E+04	73	-0.02	-1.31	1.2E+04
14	0.12	9.84	1.0E+04	29	-0.03	-2.79	1.1E+04	44	0.03	2.45	1.2E+04	59	-0.02	-1.51	1.2E+04	74	0.00	0.36	1.2E+04
15	-0.08	-6.85	1.1E+04	30	0.04	3.08	1.1E+04	45	-0.02	-1.81	1.2E+04	60	0.02	1.98	1.2E+04	75	-0.02	-1.64	1.2E+04

Figure 9: Autocorrelation Function of Differenced Series

The differenced series is now stationary and so we fitted once again an ARIMA(p,d,q) model using the Box-Jenkins approach. The best model still turned out to be the ARIMA(1,1,0) model. The forecasts and forecast errors are displayed below:

Table 4: Forecast Errors ARIMA, AZURA models

NEW X	AZURA FORECAST	ARIMA FORECAST	DENSITY NEW	AZURA error	ARIMA Error
16001	-1.40394	-1.32761	-1.32757	0.076374	3.92E-05
16002	-1.40394	-1.32757	-1.32757	0.076371	8E-07
16003	-1.40394	-1.32761	-1.32754	0.076399	6.84E-05
16004	-1.40394	-1.32757	-1.32754	0.076396	3.16E-05
16005	-1.40393	-1.32761	-1.3275	0.076433	0.000108
16006	-1.40393	-1.32757	-1.3275	0.076431	7.23E-05
16007	-1.40393	-1.32761	-1.32746	0.076468	0.000147
16008	-1.40393	-1.32757	-1.32746	0.076466	0.000113
16009	-1.40392	-1.32761	-1.32742	0.076503	0.000186
16010	-1.40392	-1.32757	-1.32742	0.0765	0.000154
16011	-1.40392	-1.32761	-1.32738	0.076538	0.000226
16012	-1.40392	-1.32757	-1.32738	0.076535	0.000194
16013	-1.40391	-1.3276	-1.32738	0.076533	0.000225
16014	-1.40391	-1.32758	-1.32735	0.07656	0.000225
16015	-1.40391	-1.3276	-1.32735	0.076557	0.000254
16016	-1.4039	-1.32758	-1.32731	0.076595	0.000266
16017	-1.4039	-1.3276	-1.32731	0.076592	0.000294
16018	-1.4039	-1.32758	-1.32727	0.07663	0.000306
16019	-1.4039	-1.3276	-1.32727	0.076627	0.000333
16020	-1.40389	-1.32758	-1.32723	0.076665	0.000347

MEAN ABSOLUTE PREDICTION ERROR:	0.07651	0.00018
STANDARD ERROR OF THE MEAN:	0.00002	0.00002

Without doubt, the ARIMA model remained the more reasonable choice for forecasting the fractal dimensions of the density of primes. This is supported by the very small mean absolute prediction error for the ARIMA forecasts.

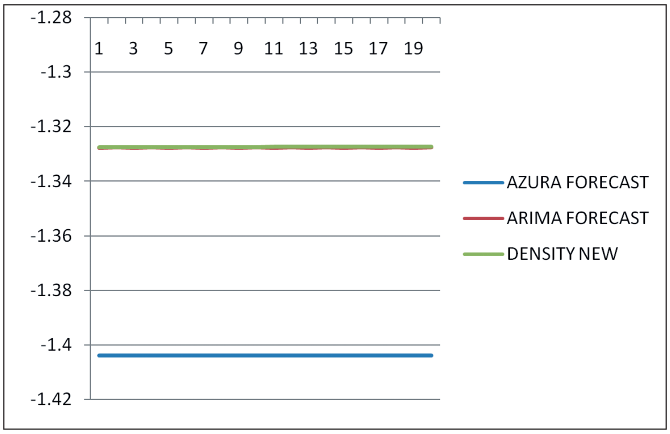
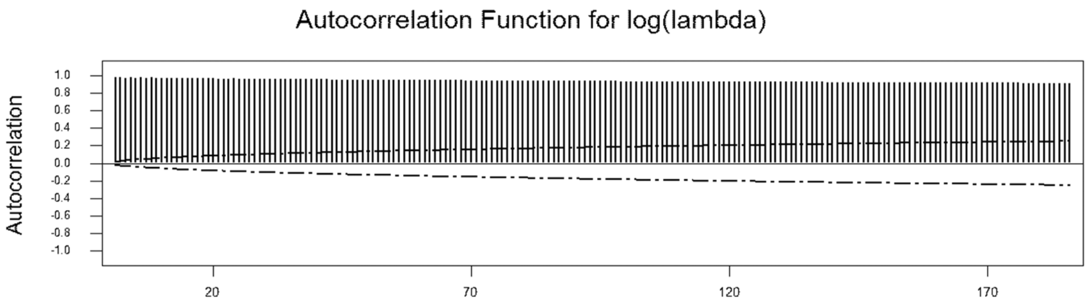


Figure 9: ARIMA AND AZURA FORECAST ERRORS (DENSITY & ARIMA COINCIDE)

Data Set 5: X = 2 to X = 20000

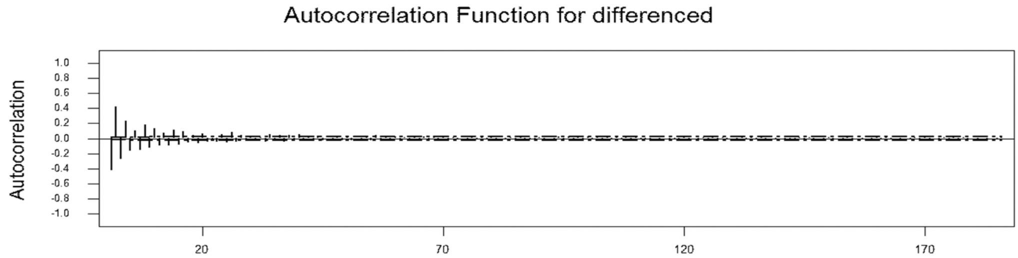
Finally, the Azura forecast function is computed for the largest data set. This is given below:
log(lambda) = 0.146 - 0.403 lnX + 0.0256 lnX-square

S = 0.04826 R-Sq = 50.3% R-Sq(adj) = 50.3%



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.98	138.73	1.9E+04	16	0.97	25.22	3.0E+05	31	0.96	17.95	5.8E+05	46	0.95	14.64	8.6E+05	61	0.95	12.67	1.1E+06
2	0.98	81.37	3.9E+04	17	0.97	24.41	3.2E+05	32	0.96	17.66	6.0E+05	47	0.95	14.49	8.8E+05	62	0.95	12.57	1.1E+06
3	0.98	62.56	5.8E+04	18	0.97	23.73	3.4E+05	33	0.96	17.38	6.2E+05	48	0.95	14.34	9.0E+05	63	0.95	12.46	1.2E+06
4	0.98	53.18	7.7E+04	19	0.97	23.08	3.6E+05	34	0.96	17.12	6.4E+05	49	0.95	14.18	9.1E+05	64	0.95	12.36	1.2E+06
5	0.97	46.75	9.6E+04	20	0.97	22.49	3.8E+05	35	0.96	16.85	6.6E+05	50	0.95	14.04	9.3E+05	65	0.95	12.26	1.2E+06
6	0.98	42.48	1.1E+05	21	0.96	21.90	4.0E+05	36	0.96	16.62	6.8E+05	51	0.95	13.89	9.5E+05	66	0.95	12.17	1.2E+06
7	0.98	39.05	1.3E+05	22	0.97	21.40	4.2E+05	37	0.96	16.38	6.9E+05	52	0.95	13.75	9.7E+05	67	0.95	12.07	1.2E+06
8	0.98	36.38	1.5E+05	23	0.97	20.93	4.3E+05	38	0.96	16.16	7.1E+05	53	0.95	13.62	9.9E+05	68	0.95	11.98	1.3E+06
9	0.97	34.04	1.7E+05	24	0.97	20.50	4.5E+05	39	0.96	15.93	7.3E+05	54	0.95	13.50	1.0E+06	69	0.95	11.89	1.3E+06
10	0.97	32.25	1.9E+05	25	0.96	20.06	4.7E+05	40	0.96	15.73	7.5E+05	55	0.95	13.37	1.0E+06	70	0.94	11.80	1.3E+06
11	0.97	30.62	2.1E+05	26	0.96	19.67	4.9E+05	41	0.96	15.52	7.7E+05	56	0.95	13.25	1.0E+06	71	0.94	11.71	1.3E+06
12	0.97	29.31	2.3E+05	27	0.96	19.26	5.1E+05	42	0.96	15.34	7.9E+05	57	0.95	13.12	1.1E+06	72	0.94	11.63	1.3E+06
13	0.97	28.10	2.5E+05	28	0.96	18.91	5.3E+05	43	0.96	15.16	8.0E+05	58	0.95	13.00	1.1E+06	73	0.94	11.55	1.3E+06
14	0.97	27.05	2.7E+05	29	0.96	18.56	5.5E+05	44	0.96	14.98	8.2E+05	59	0.95	12.88	1.1E+06	74	0.94	11.47	1.4E+06
15	0.97	26.04	2.8E+05	30	0.96	18.25	5.6E+05	45	0.95	14.80	8.4E+05	60	0.95	12.78	1.1E+06	75	0.94	11.39	1.4E+06

Figure 10: Autocorrelation Function of Raw Data



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	-0.42	-59.87	3585.00	16	0.09	8.70	1.3E+04	31	-0.04	-3.27	1.4E+04	46	0.00	0.40	1.5E+04	61	-0.02	-1.81	1.5E+04
2	0.42	50.93	7109.13	17	-0.06	-5.42	1.3E+04	32	0.01	1.10	1.4E+04	47	-0.02	-2.23	1.5E+04	62	0.01	0.92	1.5E+04
3	-0.27	-29.25	8573.30	18	0.05	4.37	1.4E+04	33	-0.04	-3.99	1.4E+04	48	0.03	2.54	1.5E+04	63	-0.02	-2.24	1.5E+04
4	0.24	24.55	9693.17	19	-0.06	-5.85	1.4E+04	34	0.05	4.80	1.4E+04	49	-0.03	-2.56	1.5E+04	64	0.03	2.44	1.5E+04
5	-0.17	-16.95	1.0E+04	20	0.07	6.30	1.4E+04	35	-0.04	-3.20	1.4E+04	50	0.04	3.21	1.5E+04	65	-0.02	-1.85	1.5E+04
6	0.11	10.73	1.0E+04	21	-0.04	-4.11	1.4E+04	36	0.04	3.91	1.4E+04	51	-0.02	-1.90	1.5E+04	66	0.03	2.60	1.5E+04
7	-0.16	-15.43	1.1E+04	22	-0.00	-0.27	1.4E+04	37	-0.04	-3.77	1.4E+04	52	0.00	0.12	1.5E+04	67	-0.02	-2.20	1.5E+04
8	0.18	17.95	1.2E+04	23	-0.05	-4.56	1.4E+04	38	0.05	4.41	1.4E+04	53	-0.02	-2.25	1.5E+04	68	0.03	3.10	1.5E+04
9	-0.13	-12.08	1.2E+04	24	0.06	5.24	1.4E+04	39	-0.04	-3.30	1.4E+04	54	0.03	2.79	1.5E+04	69	-0.02	-1.83	1.5E+04
10	0.14	13.09	1.2E+04	25	-0.06	-5.27	1.4E+04	40	0.05	4.64	1.5E+04	55	-0.03	-2.67	1.5E+04	70	0.03	2.31	1.5E+04
11	-0.09	-8.67	1.3E+04	26	0.08	7.42	1.4E+04	41	-0.03	-2.42	1.5E+04	56	0.05	4.17	1.5E+04	71	-0.02	-1.35	1.5E+04
12	0.08	7.29	1.3E+04	27	-0.05	-4.28	1.4E+04	42	0.03	2.83	1.5E+04	57	-0.02	-2.21	1.5E+04	72	0.01	1.27	1.5E+04
13	-0.10	-9.30	1.3E+04	28	0.05	4.25	1.4E+04	43	-0.03	-2.76	1.5E+04	58	0.02	1.98	1.5E+04	73	-0.02	-1.46	1.5E+04
14	0.12	11.00	1.3E+04	29	-0.03	-3.12	1.4E+04	44	0.03	2.74	1.5E+04	59	-0.02	-1.68	1.5E+04	74	0.00	0.40	1.5E+04
15	-0.08	-7.66	1.3E+04	30	0.04	3.44	1.4E+04	45	-0.02	-2.03	1.5E+04	60	0.02	2.21	1.5E+04	75	-0.02	-1.84	1.5E+04

Figure 11: Autocorrelation Function of Differenced Series

Figures 10 and 11 show that the differenced series is stationary while the raw data is non-stationary even for this larger sample size. We fitted an ARIMA(1,1,0) model to the differenced series to obtain the following forecast errors:

Table 6: Forecast Errors of Azura and ARIMA Models

NEW X	AZURA FORECAST	ARIMA FORECAST	DENSITY NEW	AZURA error	ARIMA Error
20001	-1.33428	-1.32761	-1.32757	0.006706	3.92E-05
20002	-1.33427	-1.32757	-1.32757	0.006701	8E-07
20003	-1.33427	-1.32761	-1.32754	0.006726	6.84E-05
20004	-1.33426	-1.32757	-1.32754	0.006721	3.16E-05
20005	-1.33426	-1.32761	-1.3275	0.006755	0.000108
20006	-1.33425	-1.32757	-1.3275	0.00675	7.23E-05
20007	-1.33424	-1.32761	-1.32746	0.006785	0.000147
20008	-1.33424	-1.32757	-1.32746	0.00678	0.000113
20009	-1.33423	-1.32761	-1.32742	0.006815	0.000186
20010	-1.33423	-1.32757	-1.32742	0.006809	0.000154
20011	-1.33422	-1.32761	-1.32738	0.006844	0.000226
20012	-1.33422	-1.32757	-1.32738	0.006839	0.000194
20013	-1.33421	-1.3276	-1.32738	0.006834	0.000225
20014	-1.33421	-1.32758	-1.32735	0.006859	0.000225
20015	-1.3342	-1.3276	-1.32735	0.006853	0.000254
20016	-1.3342	-1.32758	-1.32731	0.006888	0.000266
20017	-1.33419	-1.3276	-1.32731	0.006883	0.000294
20018	-1.33419	-1.32758	-1.32727	0.006918	0.000306
20019	-1.33418	-1.3276	-1.32727	0.006913	0.000333
20020	-1.33418	-1.32758	-1.32723	0.006947	0.000347

MEAN ABSOLUTE PREDICTION ERROR:	0.00682	0.00018
STANDARD ERROR OF MEAN:	0.00002	0.00002

Tabular values show that the ARIMA model is the better choice for prediction purposes.

In summary, we have demonstrated that the Azura function beats the ARIMA(1,1,0) in only one of five instances. The ARIMA model is the better option for forecasting the fractal dimension of the density of primes less or equal to a positive integer x.

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