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Original Article

On Type 2 Degenerate Poly-Frobenius-Euler Polynomials

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Abstract

Background: *This paper introduces a class of special polynomials called Type 2 degenerate poly-Frobenius-Euler polynomials, defined using the polyexponential function. Motivated by the expanding theory of degenerate versions of classical polynomials, the paper seeks to enrich the mathematical landscape by constructing generalized structures with deeper combinatorial and analytic properties.*

Methods: *The study employs the method of generating functions combined with Cauchy's rule for the product of two series to derive explicit formulas and identities, enabling systematic manipulation of series expansions. From an analytic perspective, the authors utilized the comparison test and principles of uniform convergence to establish that certain integral representations correspond to holomorphic functions.*

Results: *The researchers successfully derived explicit formulas and identities for the Type 2 degenerate poly-Frobenius-Euler polynomials. They established meaningful connections with the degenerate Stirling numbers of the first and second kinds. Furthermore, they introduced the Type 2 degenerate unipoly-poly-Frobenius-Euler polynomials, defined via the unipoly function, and thoroughly investigated their various properties, including behaviors under differentiation and integration.*

Conclusion: *The study significantly advances the theory of degenerate polynomials by constructing new polynomial families, derivation of explicit identities, and establishing analytic properties. It opens new avenues for future research by bridging classical and generalized combinatorial sequences within a robust analytic framework.*

Keywords

polyexponential functions, type 2 poly-Frobenius-Euler polynomials, unipoly functions

INTRODUCTION

Many mathematicians are actively exploring special functions and their various extensions and generalizations, including Frobenius-Euler-Genocchi Polynomials, Bivariate *(p, q)*– Bernoulli-Fibonacci Polynomials, Bivariate *(p, q)*– Bernoulli-Lucas Polynomials, Apostol-Type Frobenius-Euler Polynomials, *q* – Trigonometric Functions, the reverse Fibonacci means, *(p, q)* – Fibonacci Polynomials, and (p,q)-Lucas Polynomials (Alam et al., 2023; Elizalde & Patan, 2022; Guan et al., 2023; Rao et al., 2023; Zhang et al., 2023). These unique mathematical constructs display fascinating properties, particularly explicit formulas with practical computer modeling applications.

Special polynomials and their generating function play a crucial role in various fields of mathematics, including probability, statistics, mathematical physics, and engineering. Polynomials are particularly valuable because they can easily undergo well-established operations such as differentiation and integration, making them valuable tools for addressing real-world problems. Over recent years, many researchers have focused on the generating functions of special polynomials, examining their congruence properties, recurrence relations, computational formulas and symmetric sums (Araci & Acikgoz, 2012; Muhiuddin, Khan, & Al-Kadi, 2021).

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In recent studies, T. Kim and D. S. Kim (2018), T. Kim et al. (2015), T. Kim (2017), T. Kim and D.S. Kim (2020), T. Kim et al. (2020), Khan, Duran, et al. (2023), Khan, Alatawi, et al. (2023), Khan (2023), Khan and Kamarujjama (2023), Khan (2018), Muhiuddin, Khan, and Al-Kadi (2021), Muhiuddin et al. (2022) , and Muhiuddin, Khan, & Younis (2021), have actively explored degenerate versions of special numbers and polynomials. This approach introduces a robust framework for defining degenerate forms of special numbers and polynomials, which have significant applications. Degenerate polynomials, in particular, find utility in finite difference theory, analytic number theory, classical analysis, and statistics. Beyond these domains, special functions also emerge in communication systems, quantum mechanics, nonlinear wave propagation, electric circuit theory, and electromagnetic theory.

In this paper, we adopt the standard notations: $\mathbb{N} = \{1,2, \dots .\}$, $\mathbb{N}_0 = \{0,1,2, \dots\} = \mathbb{N} \cup \{0\}$, and $\mathbb{Z}^- = \{-1, -2, \dots\}$. Additionally, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ represents the real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

The classical Bernoulli $B_n(x)$, Euler $E_n(x)$, and Genocchi $G_n(x)$ polynomials are defined by the following generating functions:

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad |t| < 2\pi,$$
$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \qquad |t| < \pi,$$
$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \qquad |t| < \pi \tag{1.1}$$

as seen in Carlitz (1956), Eastham (1964), T. Kim and D. S. Kim (2018), T. Kim et al. (2015), and T. Kim et al. (2020).

The classical Frobenius -Euler polynomials $H_n^{(\alpha)}(x; u)$ of order α, for $u \in \mathbb{C}$ with $u \neq 1$, are given by the generating functions:

$$\left(\frac{1-u}{e^t - u}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x; u) \frac{t^n}{n!}, \tag{1.2}$$

(Araci & Acikgoz, 2012; Carlitz, 1979; Khan, 2023; Khan & Kamarujjama, 2023). In the special case where $x = 0$, $H_n^{(\alpha)}(u) = H_n^{(\alpha)}(0; u)$ are referred to as the Frobenius -Euler numbers of order α. For $\alpha = 1$, $H_n^{(1)}(x; u) = H_n(x, u)$ are called the Frobenius-Euler polynomials, and $H_n^{(\alpha)}(0; u) = h_n^{(\alpha)}(u)$ represent the Frobenius-Euler numbers of order α. Substituting $u = -1$ into (1.2) yields $H_n(x; -1) = E_m(x)$, known as the Euler polynomials (Janson, 2013; T. Kim & D. S. Kim, 2020; Muhiuddin et al., 2022; Muhiuddin, Khan, & Al-Kadi, 2021).

The polylogarithm function, for $k \in \mathbb{Z}$, is defined by:

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad |x| < 1, \tag{1.3}$$

(Eastham, 1964; Kaneko, 2025). For $k=1$, this simplifies to:

$$Li_1(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}=-\log(1-x).$$

In 2017, Kurt introduced the poly-Frobenius-Euler polynomials as:

$$\frac{(1-u)Li_k(1-e^{-t})}{t(e^t-u)}e^{xt}=\sum_{n=0}^{\infty}H_n^{(k)}(x;u)\frac{t^n}{n!} \tag{1.4}$$

When $x=0, H_n^{(k)}(u)=H_n^{(k)}(0;u)$ are referred to as the poly-Frobenius-Euler numbers.

The degenerate exponential function for nonzero $\lambda\in\mathbb{R}$ (or $\mathbb{C}$) is given by:

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}},\;\; e_\lambda(t)=(1+\lambda t)^{\frac{1}{\lambda}} \tag{1.5}$$

(D. S. Kim & T. Kim, 2019; T. Kim & D. S. Kim, 2018). This can be expanded as:

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \tag{1.6}$$

(Khan, 2018; Kurt & Simsek, 2013), where $(x)_{n,\lambda}$ is the falling factorial. As $\lambda\to 0$, $e_\lambda^x(t)$ approaches the standard exponential function e^{xt}.

Kim et al. (2015) introduced degenerate Frobenius -Euler polynomials with the generating function:

$$\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}h_{n,\lambda}(x|u)\frac{t^n}{n!}. \tag{1.7}$$

For $x=0, h_{n,\lambda}(u)=h_{n,\lambda}(0|u)$ are called degenerate Frobenius -Euler numbers.

T. Kim and D. S. Kim (2020) introduced the polyexponential function, the inverse of the polylogarithm, defined by:

$$\mathrm{Ei}_k(x)=\sum_{n=0}^{\infty}\frac{x^n}{(n-1)!\,n^k},\quad k\in\mathbb{Z}. \tag{1.8}$$

For $k=1$, this becomes:

$$\mathrm{Ei}_1(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x-1 \tag{1.9}$$

Khan (2023) further introduced degenerate poly-Euler polynomials, defined by

$$\frac{\mathrm{Ei}_k(\log(1+t))}{t(e_\lambda(t)+1)} e_\lambda^\alpha(t) = \sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \tag{1.10}$$

when $x = 0$, $E_{n,\lambda}^{(k)} = E_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Euler numbers.

The Stirling numbers of the first and second kinds are given by the following well-known relations:

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{1.11}$$

$$(x)^n = \sum_{l=0}^{n} s_2(n,l)x_l$$

From (1.11), it is easily to see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{l=0}^{\infty} S_1(n,k)\frac{t^n}{n!} \tag{1.12}$$

The degenerate Stirling numbers of the first and second kind, denoted by $S_{1,\rho}(n,k)$ and $S_{2,\rho}(n,k)$, were defined in (Kim et al., 2015; T. Kim, 2017; T. Kim & D. S. Kim, 2020; T. Kim et al., 2020) as coefficients of the following generating functions:

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_{1,\rho}(n,k)\frac{t^n}{n!}, \quad \frac{\left(e_\rho(t)-1\right)^k}{k!} = \sum_{n=0}^{\infty} S_{2,\rho}(n,k)\frac{t^n}{n!}, \tag{1.13}$$

where

$$\log_\rho\left(e_\rho(t)\right) = e_\rho(\log_\rho(t)) = t \tag{1.14}$$

When $\rho \to 0$,

$$\lim_{\rho\to 0} S_{1,\rho}(n,k) = S_1(n,k), \quad \lim_{\rho\to 0} S_{2,\rho}(n,k) = S_2(n,k).$$

In this paper, we construct the type 2 degenerate poly-Frobenius-Euler polynomials and numbers using the polyexponential function, deriving several properties and relationships for these polynomials. In the final section, we defined Type 2 degenerate unipoly-Frobenius-Euler polynomials using the unipoly function explicit expressions and properties of these polynomials.

Type 2 Degenerate Poly-Frobenius-Euler Polynomials

Let $\lambda, u \in \mathbb{C}$ with $u \neq 1$ and $k \in \mathbb{Z}$, by using the polyexponential function, we consider the type 2 degenerate poly-Frobenius-Euler polynomials are defined by means of the following generating function.

$$\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}e_\lambda^x(t)=\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}. \tag{2.1}$$

In the special case, $x=0$, $H_{n,\lambda}^{(k)}(u)=H_{n,\lambda}^{(k)}(0;u)$ are called the type 2 degenerate poly-Frobenius-Euler numbers.

For $k=1$ in (2.1), we get

$$\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t)=\sum_{n=0}^{\infty}H_{n,\lambda}(x;u)\frac{t^n}{n!}, \qquad (see\ [16]) \tag{2.2}$$

where $H_{n,\lambda}(x;u)$ are called the degenerate Frobenius-Euler polynomials. Obviously,

$$\lim_{\lambda\to\infty}\left(\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}\right)e_\lambda^x(t)=\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\ H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}$$
$$=\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e\,(t)-u)}e^x(t)=\sum_{n=0}^{\infty}H_n^{(k)}(x;u)\frac{t^n}{n!}. \tag{2.3}$$

Thus, by (2.1) and (2.2), we have

$$\lim_{\lambda\to 0}H_{n,\lambda}^{(k)}(x;u)=H_n^{(k)}(x;u), \qquad (n\geq 0) \tag{2.3}$$

where $H_n^{(k)}(x;u)$ are called the type 2 poly-Frobenius-Euler polynomials.

Theorem 2.1. *For* $n\geq 0$, *we have*

$$\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{1}{(m+1)^{k-1}}S_1(l+1,m+1)(x)_{n-l,\lambda}\frac{(1-u)^{l+1}}{l+1}$$
$$=\sum_{m=0}^{n}\binom{n}{m}H_{n-m,\lambda}^{(k)}(x;u)(1)_{m,\lambda}-uH_{n,\lambda}^{(k)}(x;u)$$

Proof. From (2.1), we have

$$\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}e_\lambda^x(t)=e_\lambda(t)\sum_{n=0}^{\infty}H_n^{(k)}(x;u)\frac{t^n}{n!}-u\sum_{n=0}^{\infty}H_n^{(k)}(x;u)\frac{t^n}{n!}$$
$$=\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}-u\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(x;u)(1)_{m,\lambda} - uH_{n,\lambda}^{(k)}(x;u) \right) \frac{t^n}{n!} \tag{2.4}$$

On the other hand,

$$\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)} e_\lambda^x(t) = \left(\sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \right) \frac{1}{t} \left(\sum_{m=1}^{\infty} \frac{(\log(1+(1-u)t))^{\,m}}{(m-1)!\,m^k} \right)$$

$$= \left(\sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \right) \frac{1}{t} \left(\sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m+1}^{\infty} S_1\,(l,m+1) \frac{(1-u)^l t^l}{l!} \right)$$

$$= \left(\sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} S_1(l+1,m+1) \frac{(1-u)^{l+1}}{l+1!} \frac{t^l}{l!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} S_1(l+1,m+1)(x)_{n-l,\lambda} \frac{(1-u)^{l+1}}{l+1!} \right) \frac{t^n}{n!} \tag{2.5}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the result.■

Theorem 2.2. *For* $n \geq 0,$ *we have*

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u) \frac{t^n}{n!} = \sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(u)(x)_{m,\lambda}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u) \frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)} \right) e_\lambda^x(t)$$

$$= \sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u) \frac{t^n}{n!} \sum_{m=0}^{\infty} (x)_{m,\lambda} \frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(u)(x)_{m,\lambda} \frac{t^n}{n!} \tag{2.6}$$

Therefore, by (2.1) and (2.6), we require at the desired result. ■

Theorem 2.3. *For* $k \in \mathbb{Z}$ *and* $n \geq 0,$ *we have*

$$H_{n,\lambda}^{(k)}(x;u) = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1,m+1)(1-u)^l}{l+1} H_{n-l,\lambda}(x;u).$$

Proof. It is proved by using (1.7), (1.11) and (2.1) that

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}\right) e_\lambda^x(t)$$

$$= \frac{e_\lambda^x(t)}{t(e_\lambda(t)-u)} \sum_{m=1}^{\infty} \frac{(\log(1+(1-u)t))^{\,m}}{(m-1)!\, m^k}$$

$$= \frac{e_\lambda^x(t)}{t(e_\lambda(t)-u)} \sum_{m=0}^{\infty} \frac{(\log(1+(1-u)t))^{\,m+1}}{(m)!\,(m+1)^k}$$

$$= \frac{e_\lambda^x(t)}{t(e_\lambda(t)-u)} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{n=m+1}^{\infty} S_1\,(n,m+1)\frac{((1-u)t)^n}{n!}$$

$$= \frac{1-u}{e_\lambda(t)-u} e_\lambda^x(t) \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{n=m}^{\infty} \frac{S_1(n+1,m+1)(1-u)^n}{n+1}\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} H_{n,\lambda}(x;u)\frac{t^n}{n!} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1,m+1)(1-u)^l}{l+1}\frac{t^l}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1,m+1)(1-u)^l}{l+1} H_{n-l,\lambda}(x;u) \right) \frac{t^n}{n!} \tag{2.7}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we complete the proof. ■

Corollary 2.4. *For* $k \in \mathbb{Z}$ *and* $n \geq 0,$ *we have*

$$H_{n,\lambda}^{(k)}(u) = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1,m+1)(1-u)^l}{l+1} H_{n-l,\lambda}(u).$$

Corollary 2.5. *For* $n \geq 0,$ *we have*

$$H_{n,\lambda}(x;u)=\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{S_1(l+1,m+1)(1-u)^l}{l+1}H_{n-l,\lambda}(x;u).$$

Corollary 2.6. *For* $n\geq 0$, *we have*

$$E_{n,\lambda}(x)=\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{S_1(l+1,m+1)2^l}{l+1}E_{n-l,\lambda}(x).$$

In particular,

$$\sum_{l=1}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{S_1(l+1,m+1)2^l}{l+1}E_{n-l,\lambda}(x)=0$$

It is well-known from Khan (2023), D. S. Kim and T. Kim (2013), and T. Kim (2017) that

$$\left(\frac{t}{\log(1+t)}\right)^r(1+t)^{x-1}=\sum_{n=0}^{\infty}B_n^{(n-r+1)}(x)\frac{t^n}{n!},\quad (r\in\mathbb{C}),\tag{2.8}$$

where $B_n^{(r)}(x)$ are called the higher-order Bernoulli polynomials which are given by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}.$$

Theorem 2.7. *For* $n\geq 0$, *we have*

$$H_{n,\lambda}^{(2)}(x;u)=\sum_{l=0}^{l}\binom{n}{l}\frac{(1-u)^l B_l^{(l)}}{l+1}H_{n-l,\lambda}(u).$$

Proof. Using (1.8), we first consider the following expression

$$\frac{d}{dx}\mathrm{Ei}_k(\log(1+(1-u)x))=\frac{d}{dx}\sum_{n=1}^{\infty}\frac{(\log(1+(1-u)x))^n}{(n+1)!\,n^k}$$

$$=\frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)}\sum_{n=1}^{\infty}\frac{(\log(1+(1-u)x))^n}{(n+1)!\,n^{k-1}}$$

$$= \frac{1-u}{(1+(1-u)x)\log(1+(1-u)x)}\mathrm{Ei}_k(\log(1+(1-u)x)) \tag{2.9}$$

From (2.9), $k \geq 1$, we have

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(u)\frac{x^n}{n!} = \frac{(1-u)^{k-1}}{x(e_\lambda(x)-u)}\underbrace{\int_0^x \frac{1}{(1+(1-u)t)\log(1+(1-u)t)}}_{(k-1)-times}$$

$$\times \underbrace{\int_0^t \frac{1}{(1+(1-u)t)\log(1+(1-u)t)} \cdots \int_0^t}\frac{1}{(1+(1-u)t)\log(1+(1-u)t)}dtdt \ldots dt$$

Hence, we require

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(2)}(u)\frac{x^n}{n!} = \frac{(1-u)}{x(e_\lambda(x)-u)}\int_0^x \frac{(1-u)tdt}{(1+(1-u)t)\log(1+(1-u)t)} \tag{2.10}$$

$$= \frac{(1-u)}{x(e_\lambda(x)-u)}\int_0^x \sum_{n=0}^{\infty}(1-u)^n B_n^{(n)}\frac{t^n}{n!}dt$$

$$= \frac{(1-u)}{x(e_\lambda(x)-u)}\sum_{n=0}^{\infty}\frac{(1-u)^n B_n^{(n)}}{n+1}\frac{x^n}{n!}$$

$$= \left(\sum_{n=0}^{\infty} H_{n,\lambda}(u)\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(1-u)^n B_n^{(n)}}{n+1}\frac{x^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{l}\binom{n}{l}\frac{(1-u)^l B_l^{(l)}}{l+1}H_{n-l}(u)\right)\frac{x^n}{n!} \tag{2.11}$$

By (2.10) and (2.11), we obtain at the desired result. Thus, we complete the proof. ■

Theorem 2.8. *Let* $k \geq 1$ *and* $m \in \mathbb{N} \cup \{0\}, s \in \mathbb{C}$, *we have*

$$\chi_{k,u,v}(-m) = (1-u)^{-m-1}(-1)^m H_{m,\lambda}^{(k)}(u)$$

Proof. Let $k \geq 1$, be an integer. For $s \in \mathbb{C}$, we define the function $\chi_{k,u,v}(s)$ as

$$\chi_{k,u,v}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{z^{s-1}}{z(e_v(z) - u)} \operatorname{Ei}_k(\log(1 + (1-u)z))\ dz. \tag{2.12}$$

Note that the integrand is given by:

$$\frac{z^{s-1}}{z(e_v(z) - u)} Ei_k(\log(1 + (1-u)z)),$$

where:

- $\Gamma(s)$ is the Gamma function, holomorphic for $s \neq 0, -1, -2, \ldots$,
- $z^{s-1} = e^{(s-1)\log(z)}$, where $\log(z)$ is the principal branch of the logarithm,
- $e_v(z)$ is the exponential-like function, and $e_v(z) - u \neq 0$,
- Ei_k involves a series expansion of logarithmic terms.

As $z \to 0^+$:

- z^{s-1} behaves as $z^{\Re(s)-1}$,
- For small $z, e_v(z) \sim 1$, so $e_v(z) - u$ remains finite for $u \neq 1$.
- $Ei_k(\log(1 + (1-u)z))$ is well behaved for small z, as $\log(1 + (1-u)z) \sim (1-u)z$.

Thus, near $z = 0$, the integrand behaves as $z^{\Re(s)-1}$, which converges if $\Re(s) > 0$.

As $z \to \infty$:

- For large $z, e_v(z)$ grows exponentially, ensuring $e_v(z) - u$ does not vanish.
- The term $Ei_k(\log(1 + (1-u)z))$ involves higher-order logarithmic growth, but its growth is tempered by the denominator $e_v(z) - u$.

For large z, the integrand behaves approximately as:

$$\frac{z^{\Re(s)-1}}{z e_v(z)} \sim \frac{z^{\Re(s)-2}}{e_v(z)}.$$

The exponential growth of $e_v(z)$ dominates any polynomial growth in $z^{\Re(s)-2}$ ensuring converges as $z \to \infty$. To apply the Comparison Test, compare the magnitude of the integrand with a simpler, absolutely convergent integral. For $\Re(s) > 0$, the magnitude of the integrand can be bounded as:

$$\left| \frac{z^{s-1}}{z(e_v(z) - u)} Ei_k(\log(1 + (1-u)z)) \right| \leq C \frac{|z|^{\Re(s)-2}}{|e_v(z)|}$$

where C is a constant accounting for the boundedness of Ei_k and $e_v(z) - u$. The integral of this bound converges absolutely for $\Re(s) > 0$. Moreover one may observe the following holomorphic dependence on s:

- The factor $z^{s-1} = e^{(s-1)\log(z)}$ depends holomorphically on s.
- The integral converges uniformly for $\Re(s) > 0$, ensuring the holomorphicity of the integral as a function of s.

By the Comparison Test and the uniform convergence of the integral for $\Re(s) > 0, \chi_{k,u,v}(s)$ is holomorphic in this region. The Gamma function $\Gamma(s)$ introduces additional singularities at $s = , -1, -2, \ldots$, but $\chi_{k,u,v}(s)$ remains holomorphic where s avoids these poles.

From (2.12), we note that

$$\chi_{k,u,v}(s) = \frac{(1-u)^{s-1}}{\Gamma(s)} \int_0^\infty \frac{z^{s-1}}{z(e_v(z)-u)} Ei_k(\log(1+(1-u)z))\ dz$$

$$= \frac{(1-u)^{s-1}}{\Gamma(s)} \int_0^1 \frac{z^{s-1}}{z(e_v(z)-u)} Ei_k(\log(1+(1-u)z))\ dz$$

$$+ \frac{(1-u)^{s-1}}{\Gamma(s)} \int_1^\infty \frac{z^{s-1}}{z(e_v(z)-u)} Ei_k(\log(1+(1-u)z))\ dz. \qquad (2.13)$$

The second integral converges absolutely for any $s \in \mathbb{C}$ and hence, the second term on the right-hand side vanishes at non-positive integers. That is,

$$\lim_{s\to -m} \left| \frac{(1-u)^{s-1}}{\Gamma(s)} \int_1^\infty \frac{z^{s-1}}{z(e_v(z)-u)} Ei_k(\log(1+(1-u)z))\ dz \right|$$

$$\leq \frac{(1-u)^{-m-1}}{\Gamma(-\mathrm{m})} M = 0, \qquad (2.14)$$

since

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

On the other hand, for $\Re(s) > 0$, the first integral in (2.13) can be written as

$$\frac{(1-u)^{s-1}}{\Gamma(s)} \int_0^1 \frac{z^{s-1}}{z(e_v(z)-u)} Ei_k(\log(1+(1-u)z))\ dz$$

$$= \frac{(1-u)^{s-1}}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{H_{n,v}^{(k)}(u)}{n!} \int_0^1 z^{n+s-1} dz$$

$$= \frac{(1-u)^{s-1}}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{H_{n,v}^{(k)}(u)}{n!} \frac{1}{n+s} \tag{2.15}$$

which defines an entire function s. Thus, we may include that $\chi_{k,u,v}(s)$ can be continued to an entire function of s. Further, from (2.14) and (2.15), we obtain

$$\chi_{k,u,v}(-m) = \lim_{s \to -m} \frac{(1-u)^{s-1}}{\Gamma(s)} \int_0^1 \frac{z^{s-1}}{z(e_v(z)-u)} Ei_k(\log(1+(1-u)z))\ dz$$

$$= \lim_{s \to -m} \frac{(1-u)^{s-1}}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{H_{n,\lambda}^{(k)}(u)}{s+n} \frac{1}{n!}$$

$$= \cdots + 0 + \cdots + 0 + \lim_{s \to -m} \frac{(1-u)^{s-1}}{\Gamma(s)} \frac{1}{s+m} \frac{H_{m,v}^{(k)}(u)}{m!} + 0 + 0 + \cdots$$

$$= \lim_{s \to -m} \frac{(1-u)^{s-1}}{\Gamma(s)} \frac{\Gamma(1-s)\sin(\pi s)}{\pi} \frac{H_{m,v}^{(k)}(u)}{m!}$$

$$= (1-u)^{-m-1}\Gamma(1+m)\cos(\pi m) \frac{H_{m,v}^{(k)}(u)}{m!} \tag{2.16}$$

$$= (1-u)^{-m-1}(-1)^m H_{m,\lambda}^{(k)}(u)$$

Thus, we complete the proof of this theorem. ∎

Theorem 2.9. *For* $n \geq 0$, *we have*

$$H_{n,\lambda}^{(k)}(x+y;u) = \sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(x;u)(y)_{m,\lambda}.$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x+y;u) \frac{t^n}{n!} = \frac{Ei_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)} e_\lambda^{x+y}(t)$$

$$= \left(\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x;u) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} (y)_{m,\lambda} \frac{t^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(x;u)(y)_{m,\lambda} \right) \frac{t^n}{n!} \tag{2.17}$$

Comparing the coefficients on both sides, we get the result.■

Corollary 2.10. *For* $n \geq 0$, *we have*

$$H_{n,\lambda}^{(k)}(x+1;u)=\sum_{m=0}^{n}\binom{n}{m}H_{n-m,\lambda}^{(k)}(x;u)(1)_{m,\lambda}.$$

Proof. By (2.1), we observe that

$$\sum_{n=0}^{\infty}\left[H_{n,\lambda}^{(k)}(x+1;u)-H_{n,\lambda}^{(k)}(x;u)\right]\frac{t^n}{n!}=\left(\frac{Ei_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}\right)e_\lambda^x(t)[e_\lambda(t)-1]$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}H_{n-m,\lambda}^{(k)}(x;u)(1)_{m,\lambda}\frac{t^n}{n!}-\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}. \tag{2.18}$$

Comparing coefficients of both sides, we get the desired result.■

Theorem 2.11. *For* $n \geq 0$, *we have*

$$H_{n,\lambda}^{(k)}(x;u)=\sum_{m=0}^{n}\sum_{q=0}^{m}\binom{n}{m}(x)_q\,S_{2,\lambda}(m,q)\,H_{n-m,\lambda}^{(k)}(u)$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(x;u)\frac{t^n}{n!}=\left(\frac{Ei_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}\right)e_\lambda^x(t)$$

$$=\left(\frac{Ei_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}\right)[e_\lambda(t)-1+1]^x$$

$$=\left(\frac{Ei_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}\right)\left(\sum_{q=0}^{\infty}(x)_q\sum_{l=q}^{\infty}S_{2,\lambda}(l,q)\frac{t^l}{l!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{q=0}^{m}\binom{n}{m}(x)_q\,S_{2,\lambda}(m,q)H_{n-m,\lambda}^{(k)}(u)\right)\frac{t^n}{n!} \tag{2.19}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the desired result.■

Theorem 2.12. *For* $n \geq 0$, *we have*

$$H_{n,\lambda}^{(k)}(x+\alpha|u)=\sum_{l=0}^{n}\sum_{q=0}^{l}\binom{n}{l}(x)_q\,S_{2,\lambda}(l,q)H_{n-l,\lambda}^{(k)}(\alpha;u)$$

Proof. Replacing x by $x+\alpha$ in (2.1), we have

$$\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(x+\alpha;u)\frac{t^n}{n!}=\left(\frac{Ei_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}\right)e_\lambda^{x+\alpha}(t)$$

$$=\left(\frac{Ei_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}e_\lambda^{x+\alpha}(t)\right)\left(\sum_{q=0}^{\infty}(x)_q\sum_{l=q}^{\infty}S_{2,\lambda}(l,q)\frac{t^l}{l!}\right)$$

$$=\left(\sum_{n=0}^{\infty}H_{n,\lambda}^{(k)}(\alpha;u)\frac{t^n}{n!}\right)\left(\sum_{q=0}^{\infty}(x)_q\sum_{l=q}^{\infty}S_{2,\lambda}(l,q)\frac{t^l}{l!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{q=0}^{l}\binom{n}{l}(x)_q\,S_{2,\lambda}(l,q)H_{n-l,\lambda}^{(k)}(\alpha;u)\right)\frac{t^n}{n!} \tag{2.20}$$

Therefore, by (2.1) and (2.20), we obtain the result.■

Type 2 degenerate unipoly-Frobenius-Euler polynomials

Let p be any arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. Kurt (2017) defined the unipoly function attached to polynomials $p(x)$ by

$$u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)}{n^k}x^n,(k\in\mathbb{Z}). \tag{3.1}$$

Moreover,

$$u_k(x|1)=\sum_{n=1}^{\infty}\frac{x^n}{n^k},(see\ [7,15]). \tag{3.2}$$

is the ordinary polylogarithm function.

By using (3.1), we define the type 2 degenerate unipoly-Frobenius-Euler polynomials by

$$\frac{u_k(\log(1+(1-u)t)|p)}{t(e_\lambda(t)-u)}e_\lambda^x(t)=\sum_{n=0}^{\infty}H_{n,\lambda,p}^{(k)}(x;u)\frac{t^n}{n!}. \tag{3.3}$$

In the case when $x=0$, $H_{n,\lambda,p}^{(k)}(u)=H_{n,\lambda,p}^{(k)}(0;u)$ are called the type 2 degenerate unipoly-Frobenius-Euler numbers. Let us take $p(n)=\frac{1}{\Gamma(n)}$. Then, we have

$$\sum_{n=0}^{\infty}H_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x;u)\frac{t^n}{n!}=\frac{u_k(\log(1+(1-u)t)|\frac{1}{\Gamma})}{t(e_\lambda(t)-u)}e_\lambda^x(t)$$

$$=\frac{1}{t(e_\lambda(t)-u)}e_\lambda^x(t)\sum_{m=1}^{\infty}\frac{(\log(1+(1-u)t))^m}{(m+1)!\,m^k}$$

$$=\frac{Ei_k(\log(1+(1-u)t))}{t(e_\lambda(t)-u)}e_\lambda^x(t)$$

$$=\sum_{n=0}^{\infty}H_{n,\lambda,}^{(k)}(x;u)\frac{t^n}{n!} \tag{3.4}$$

Thus, we have

$$H_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(x;u)=H_{n,\lambda,}^{(k)}(x;u)$$

Theorem 3.1. *Let $n\in\mathbb{N}$ and $k\in\mathbb{Z}$. Then we have*

$$H_{n,\lambda,p}^{(k)}(x;u)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}(1-u)^l H_{n-l,\lambda}(u) \tag{3.5}$$

In particular,

$$H_{n,\lambda,\frac{1}{\Gamma}}^{(k)}(u)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{(m+1)}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}(1-u)^l H_{n-l,\lambda}(u) \tag{3.6}$$

Proof. From (3.3), we get

$$\sum_{n=0}^{\infty}H_{n,\lambda,p}^{(k)}(x;u)\frac{t^n}{n!}=\frac{u_k(\log(1+(1-u)t)|p)}{t(e_\lambda(t)-u)}$$

$$=\frac{1}{t(e_\lambda(t)-u)}\sum_{m=1}^{\infty}\frac{p(m)}{m^k}(\log(1+(1-u)t))^m$$

$$= \frac{1}{t(e_\lambda(t)-u)} \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m+1}^{\infty} S_1(l, m+1) \frac{\left((1-u)t\right)^l}{l!}$$
$$= \frac{(1-u)t}{t(e_\lambda(t)-u)} \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{m=l}^{\infty} \frac{S_1(l+1, m+1)}{l+1} (1-u)^l \frac{t^l}{l!}$$

$$= \left(\sum_{n=0}^{\infty} H_{n,\lambda}(u) \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{m=l}^{\infty} \frac{S_1(l+1, m+1)}{l+1} (1-u)^l \frac{t^l}{l!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_1(l+1, m+1)}{l+1} (1-u)^l H_{n-l,\lambda}(u)\right) \frac{t^n}{n!} \tag{3.7}$$

Therefore, by comparing the coefficients on both sides of (3.7), we obtain the result.■

Theorem 3.2. *Let* $n \geq 0$ *and* $k \in \mathbb{Z}$. *Then we have*

$$H_{n,\lambda,p}^{(k)}(x; u) = \sum_{m=0}^{n} \sum_{q=0}^{m} \binom{n}{m} (x)_q \, S_{2,\lambda}(m, q) H_{n-m,\lambda,p}^{(k)}(u). \tag{3.8}$$

Proof. Recalling from (3.3) that

$$\sum_{n=0}^{\infty} H_{n,\lambda,p}^{(k)}(x; u) \frac{t^n}{n!} = \left(\frac{u_k(\log(1+(1-u)t)|p)}{t(e_\lambda(t)-u)}\right) e_\lambda^x(t)$$

$$= \left(\frac{u_k(\log(1+(1-u)t)|p)}{t(e_\lambda(t)-u)}\right) [e_\lambda(t) - 1 + 1]^x$$

$$= \left(\frac{u_k(\log(1+(1-u)t)|p)}{t(e_\lambda(t)-u)}\right) \left(\sum_{q=0}^{\infty} (x)_q \sum_{l=q}^{\infty} S_{2,\lambda}(l, q) \frac{t^l}{l!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{q=0}^{m} \binom{n}{m} (x)_q \, S_{2,\lambda}(m, q) H_{n-m,\lambda,p}^{(k)}(u)\right) \frac{t^n}{n!} \tag{3.9}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the desired result.■

Theorem 3.2. *Let $n \geq 0$ and $k \in \mathbb{Z}$. Then we have*

$$H_{n,\lambda,p}^{(k)}(x;u) = \sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda,p}^{(k)}(u)(x)_{m,\lambda} \tag{3.10}$$

Proof. It is proved by using (3.3) that

$$\sum_{n=0}^{\infty} H_{n,\lambda,p}^{(k)}(x;u)\frac{t^n}{n!} = \left(\frac{u_k(\log(1+(1-u)t)|p)}{t(e_\lambda(t)-u)}\right)e_\lambda^x(t)$$

$$= \left(\sum_{n=0}^{\infty} H_{n,\lambda,p}^{(k)}(u)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} H_{n-m,\lambda,p}^{(k)}(u)(x)_{m,\lambda}\right)\frac{t^n}{n!} \tag{3.11}$$

By comparing the coefficients of (3.11) on both sides, we obtain the result.■

CONCLUSION

Motivated by the definition of the degenerate poly-Bernoulli polynomials introduced by T. Kim et al. (2015), in the present paper, the researchers considered a class of new generating function for the degenerate Frobenius-Euler polynomials, called the type 2 degenerate poly-Frobenius-Euler polynomials, using the polyexponential function. Then, it derived some useful relations and properties. It was shown that the type 2 degenerate poly-Frobenius-Euler polynomials equal a linear combination of the degenerate Frobenius-Euler polynomials and Stirlings numbers of the first and second kind. In a special case, a relation was given between the type 2 degenerate Frobenius-Euler polynomials and Bernoulli polynomials of order n. Moreover, inspired by the definition of unipoly-Bernoulli polynomials introduced by T. Kim et al. (2020), the researchers have introduced the type 2 degenerate unipoly-Frobenius-Euler polynomials using unipoly function and given multifarious properties, including degenerate Stirling numbers of the second kind and degenerate Frobenius-Euler polynomials.

Author Contributions

Khan, R. Corcino and C. Corcino: Conceptualization, Methodology, Software; **Khan:** Writing-Original draft preparation; **Khan, R. Corcino and C. Corcino:** Visualization, Investigation; **R. Corcino:** Supervision; **R. Corcino and C. Corcino:** Software, Validation; **R. Corcino and C. Corcino:** Writing-Reviewing and Editing.

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Declaration of Artificial Intelligence Use

In this work, the author(s) did not utilize any artificial intelligence (AI) tools and methodologies in the preparation and development of the paper.

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