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Approximations of Apostol-Tangent Polynomials of Complex Order with Parameters *a*, *b*, and *c*

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Abstract

This paper presents new approximation formulas for the tangent polynomials and Apostol-tangent polynomials of complex order, specifically for large values of n. These polynomials are parameterized by a,b, and c. The derivation of these formulas is accomplished through contour integration techniques, where the contour is carefully selected to avoid branch cuts introduced by the presence of multiple singularities within the integration path. The analysis includes a detailed computation of the singularities associated with the generating functions used in this process, ensuring the accuracy and rigor of the derived formulas. Additionally, the paper provides corollary results that reinforce and affirm the newly established formulas, offering a comprehensive understanding of the behavior of these polynomials under specified conditions.

Keywords

Asymptotic approximation, tangent polynomials, Apostol-tangent polynomials

INTRODUCTION

Numerous mathematicians are actively engaged in the exploration of special functions and various hybrid variants, such as Frobenius-Euler-Genocchi Polynomials, Bivariate (p,q)-Bernoulli-Fibonacci Polynomials, Bivariate (p,q)-Bernoulli-Lucas Polynomials, Apostol-Type Frobenius-Euler Polynomials, q-Trigonometric Functions, Fibonacci sequence, (p,q)-Fibonacci, and (p,q)-Lucas Polynomials, particularly in conjunction with Changhee Numbers (Alam et al., 2023; Elizalde & Patan, 2022; Guan et al., 2023; Rao et al., 2023; Zhang et al., 2023). These distinct mathematical constructs exhibit intriguing properties, notably explicit formulas that find practical applications in computer modeling.

The polynomials that will be considered are given by the generating functions

$$\sum_{n=0}^{\infty} T_n^{(\mu)}(x;a,b,c)\frac{t^n}{n!} = \left(\frac{2}{b^{2t}+a^t}\right)^{\mu} c^{xt}$$
$$= \left(\frac{2}{e^{2t\ln b}+e^{t\ln a}}\right)^{\mu} e^{xt\ln c}, \quad |t| < \frac{\pi}{B} \qquad (1)$$

and

$$\sum_{n=0}^{\infty} T_n^{(\mu)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \left(\frac{2}{\lambda b^{2t}+a^t}\right)^{\mu} c^{xt}$$
$$= \left(\frac{2}{\lambda e^{2t\ln b}+e^{t\ln a}}\right)^{\mu} e^{xt\ln c}, \quad |t| < \frac{|\pm\pi i - 2\xi|}{B} \qquad (2)$$

where $T_n^{(\mu)}(x;a,b,c)$denotes the tangent polynomials of order μ and $T_n^{(\mu)}(x;\lambda,a,b,c)$ denote the Apostol-tangent polynomials of order μ with $\mu \in \mathbb{C}, \lambda = e^{2\xi}, B = 2\ln b - \ln a > 0$ and a, b, c are positive real numbers.

Asymptotic approximations of Bernoulli polynomials, Euler Polynomials, and Genocchi polynomials of complex order were obtained using contour integration (López & Temme, 2010). Asymptotic approximations of Apostol-Bernoulli polynomials, Apostol-Euler Polynomials, Apostol-Genocchi polynomials, and Apostol-tangent polynomials were derived using Fourier series and ordering of poles of the generating function (Corcino et al., 2022). Asymptotics for complex order tangent and Apostol-tangent were obtained in Corcino et al. (2023) following the method in López and Temme (2010). The 2-variable q-generalized tangent-Apostol-type polynomials, a new class of q-hybrid special polynomials, were studied in Yasmin and Muhyi (2021). Interesting properties for a new generalization for tangent polynomials were derived in Bildirici et al. (2014). Mathematicians were attracted to work on tangent polynomials because of their applications in the field of mathematics and physics (Ryoo, 2013a; Ryoo, 2013b). We follow Yasmin and Muhyi (2021) in the use of the small letter tangent polynomials.

This paper will investigate the method used in Corcino et al. (2023) and López and Temme (2010) to find asymptotic formulas of tangent and Apostol-tangent polynomials of complex order μ with parameters a, b, and c.

Asymptotic expansions of tangent polynomials of complex order μ with parameters a, b, c

The singularities of the generating function in (1) can be obtained by setting the denominator equal to zero. This is done as follows:

$$b^{2t} + a^t = 0e^{2t\ln b} + 3^{t\ln a} = 0, \exp\{(2t\ln b - t\ln a)\} = -1$$

$$t(2\ln b - \ln a) = (2n+1)\pi i t = \frac{(2n+1)\pi i}{2\ln b - \ln a}.$$

The singularities of (1) nearest to the origin are $\pm\frac{\pi i}{B}$, where $B = 2\ln b - \ln a$. The main asymptotic contribution is derived from the singularities at t_0 and t_{-1}. The Cauchy Integral Formula in complex analysis will be applied to obtain an integral representation of the polynomials. For more discussion on the Cauchy Integral Formula, see Churchill et al. (1976). The integration will be done around a circle C_1 about 0 with radius $|t_0| + \epsilon, \epsilon > 0$, avoiding the branch cuts along the lines $y = \frac{\pi i}{B}$ and $y = -\frac{\pi i}{B}$ of (1). See Figure 1 to visualize the contour of integration. In Figure 1, B is assumed to be greater than 2 so that $t_0 = \frac{\pi i}{B}$ below $\frac{\pi i}{2}$ and $t_{-1} = -\frac{\pi i}{B}$ is above $-\frac{\pi i}{2}$. Applying Cauchy Integral Formula with C_1 as the contour of integration yields,

$$\frac{n!}{2\pi i}\int_{C_1}\left(\frac{2}{e^{2t\ln b}+e^{t\ln a}}\right)^{\mu} e^{xt\ln c}\frac{dt}{t^{n+1}} = Res[f(t),0], \qquad (3)$$

$$Res[f(t),0]=\frac{n!}{2\pi i}\left(\int_{C^*}f(t)dt+\int_{L_-}f(t)dt\int_{L_+}f(t)dt\right). \quad (4)$$

By the principle of deformation of paths,

$$Res[f(t),0]=\frac{n!}{2\pi i}\int_{C}\left(\frac{2}{e^{2t\ln b}+e^{t\ln a}}\right)^{\mu}e^{xt\ln c}\frac{dt}{t^{n+1}}=T_n^{(\mu)}(x;a,b,c), \quad (5)$$

where C is a circle with a radius less than π/B. Then (4) and (5) yield

$$T_n^{(\mu)}(x;a,b,c)=\frac{n!}{2\pi i}\left(\int_{C^*}f(t)dt+\int_{L_-}f(t)dt\int_{L_+}f(t)dt\right). \quad (6)$$

Lemma 2.1. *As $n\to+\infty$, the integral along C^* is $\mathcal{O}(\pi^{-n})$. That is,*

$$\int_{C^*}f(t)dt=\mathcal{O}(\pi^{-n})$$

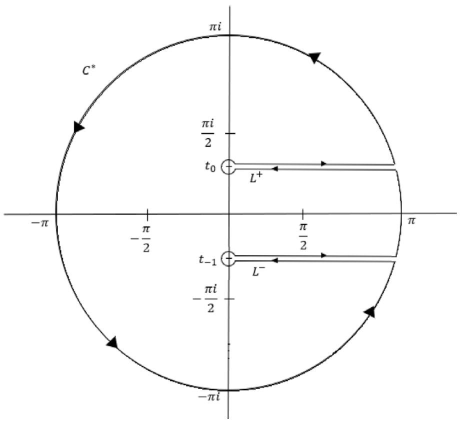

Figure 1. *Contour for tangent polynomials of complex order μ with parameters a, b and c*

Proof. Let

$$f(t)=\left(\frac{2}{e^{2t\ln b}+e^{t\ln a}}\right)^{\mu}\frac{e^{xt\ln c}}{t^{n+1}},$$

where μ and x are fixed complex numbers. Let $\max\{|\mu|,|x|\}\leq K$ and there is an $M>0$ such that

$$\left|\frac{c^t}{b^{2t}+a^t}\right|<M.$$

Then

$$|f(t)|\leq\frac{2^K}{|e^{2t\ln b}+e^{t\ln a}|}\cdot\frac{|e^{t\ln c}|^K}{\pi^{n+1}}<\frac{(2M)^K}{\pi^{n+1}}. \quad (7)$$

RECOLETOS MULTIDISCIPLINARY RESEARCH JOURNAL

Thus,

$$\left|\int_{C^*} f(t)dt\right| \le \frac{(2M)^K}{\pi^{n+1}} \int_{C^*} |dt| = \frac{(2M)^K \pi}{\pi^{n+1}} = O(\pi^{-n}). \tag{8}$$

∎

Consequently, we have the following theorem.

Theorem 2.2. *As* $n \to +\infty, \mu, x$ *are fixed complex numbers, and* $\beta = (2D - \mu - n - 2)\frac{\pi}{2}$ *where* $D = \frac{x \ln c - \mu \ln a}{B}$ *and* $B = 2 \ln b - \ln a$ *,* $< 1 - \mu >_k$ *denoted the kth falling factorial,*

$$T_n^{(\mu)}(x; a, b, c) \sim \frac{n!\, 2^{\mu+1} B^n n^{\mu-1}}{\pi^{n+\mu}\, \Gamma(\mu)} \left\{\cos\beta \sum_{k=0}^{\infty} \frac{\langle 1-\mu\rangle_k}{n^k} Re(F_k) - \sin\beta \sum_{k=0}^{\infty} \frac{\langle 1-\mu\rangle_k}{n^k} Im(F_k)\right\} \tag{9}$$

Proof. For evaluating the integrals along the specified contours, denote I_+ as the integral along the contour L_+ and I_- as the integral along the contour L_-. We begin by calculating I_+, which is expressed as follows:

$$I_+ = \frac{n!}{2\pi i} \int_{L_+} \frac{2^\mu e^{xt \ln c}}{(e^{2t \ln b} + e^{t \ln a})^\mu} \frac{dt}{t^{n+1}},$$

Here, the integrand involves a complex exponential function modified by parameters a, b, and c, and the contour L_+ is chosen to encapsulate the singularities of the integrand in a manner conducive to residue calculus. The factor t^{n+1} in the denominator is critical, as it governs the order of the pole at $t = 0$, which directly influences the outcome of the integral. The next steps involve evaluating this integral using techniques from complex analysis, particularly focusing on the behavior of the integrand along the contour L_+ Now, let $t = \frac{\pi i e^s}{B}$. Then $dt = \frac{\pi i e^s}{B} ds$ and

$$I_+ = \frac{n!\, 2^\mu}{2\pi i} \int_{C_+} \frac{e^{\frac{\pi i x e^s \ln c}{B}}}{\left(e^{\frac{2\pi i\, e^s \ln b}{B}} + e^{\frac{\pi i e^s \ln a}{B}}\right)^\mu} \frac{\frac{\pi i\, e^s}{B} ds}{\left(\frac{\pi i\, e^s}{B}\right)^{n+1}} = \frac{n!\, 2^{\mu-1} B^n}{(\pi i)^{n+1}} \int_{C_+} \frac{e^{\frac{\pi i x e^s \ln c}{B}}\, ds}{\left(e^{\frac{2\pi i\, e^s \ln b}{B}} + e^{\frac{\pi i e^s \ln a}{B}}\right)^\mu e^{ns}}.$$

Multiplying the last array with $(\pi i)^\mu (\pi i)^{-\mu}, e^{-\frac{\pi i\,(x \ln c - \mu \ln a)}{B}} e^{\frac{\pi i\,(x \ln c - \mu \ln a)}{B}}$ and since $e^{\pi i} = -1$, we have

$$\begin{aligned} I_+ &= \frac{n!\, 2^{\mu-1} B^n}{(\pi i)^{n+1}} \int_{C_+} \frac{e^{\frac{\pi i x e^s \ln c}{B}} (\pi i)^\mu (\pi i)^{-\mu}\, e^{-\frac{\pi i\,(x \ln c - \mu \ln a)}{B}} e^{\frac{\pi i\,(x \ln c - \mu \ln a)}{B}}\, ds}{\left[e^{\frac{\pi i\, e^s \ln a}{B}} e^{\pi i} \left(e^{\frac{2\pi i\, e^s \ln b}{B}} e^{-\frac{\pi i\, e^s \ln a}{B}} e^{-\pi i} - 1\right)\right]^\mu e^{ns}} \\ &= \frac{n!\, 2^{\mu-1} B^n}{(\pi i)^{\mu+n+1} e^{\pi i \mu}} \int_{C_+} \frac{(\pi i)^\mu e^{\frac{\pi i x e^s \ln c}{B}} e^{-\frac{\pi i\, e^s \ln a}{B}} e^{-\frac{\pi i\,(x \ln c - \mu \ln a)}{B}} e^{\frac{\pi i\,(x \ln c - \mu \ln a)}{B}}\, ds}{\left(e^{\frac{\pi i\, e^s (2 \ln b - \ln a)}{B} - \pi i} - 1\right)^\mu e^{ns}} \\ &= \frac{n!\, 2^{\mu-1} B^n}{(\pi i)^{\mu+n+1} e^{\pi i \mu} e^{-\frac{\pi i\,(x \ln c - \mu \ln a)}{B}}} \int_{C_+} \frac{(\pi i)^\mu e^{\frac{\pi i (x \ln c - \mu \ln a)}{B}(e^s - 1)}\, ds}{(e^{\pi i\,(e^s - 1)} - 1)^\mu e^{ns}}, \end{aligned}$$

where C_+ is the image of L_+ under the transformation $t = \frac{\pi i e^s}{B}$. C_+ is the contour that encircles the origin in the clockwise direction. Let $v = \pi i(e^s - 1)$ and $D = \frac{x \ln c - \mu \ln a}{B}$, then the last array will become

$$I_+ = \frac{n!\, 2^{\mu-1} B^n}{(\pi i)^{\mu+n+1} e^{(\mu - D)\pi i}} \int_{C_+} \frac{(\pi i)^\mu e^{Dv}\, ds}{(e^v - 1)^\mu e^{ns}} \tag{10}$$

Multiplying $s^{-\mu}\, s^{\mu}$ in equation (10), we have

$$I_+ = \frac{n!\,2^{\mu-1}B^n}{(\pi i)^{\mu+n+1}e^{(\mu-D)\pi i}} \int_{C_+} \frac{(\pi i)^{\mu} e^{Dv} s^{-\mu}\, s^{\mu}\, ds}{(e^{v}-1)^{\mu} e^{ns}}$$

$$= \frac{n!\,2^{\mu-1}B^n}{(\pi i)^{\mu+n+1}e^{(\mu-D)\pi i}} \int_{C_+} \left(\frac{\pi i s}{e^{v}-1}\right)^{\mu} e^{Dv}\, e^{-sn}\, s^{-\mu}\, ds \tag{11}$$

$$= \frac{n!\,2^{\mu-1}B^n}{(\pi i)^{\mu+n+1}e^{(\mu-D)\pi i}} \int_{C_+} F(s)\, e^{-sn}\, s^{-\mu}\, ds \tag{12}$$

where

$$F(s) = \left(\frac{\pi i s}{e^{v}-1}\right)^{\mu} e^{Dv}. \tag{13}$$

Note that applying L'Hospital's rule, $F(0) = 1$. To obtain an asymptotic expansion, we apply Watson's lemma for loop integrals and then expand,

$$F(s) = \sum_{k=0}^{\infty} F_k s^k. \tag{14}$$

Substituting (14) to (12), then I_+ becomes

$$I_+ = \frac{n!\,2^{\mu-1}B^n}{(\pi i)^{\mu+n+1}e^{(\mu-D)\pi i}} \int_{C_+} \sum_{k=0}^{\infty} F_k s^k\; e^{-sn}\, s^{-\mu}\, ds$$

$$= \frac{n!\,2^{\mu-1}B^n}{(\pi i)^{\mu+n+1}e^{(\mu-D)\pi i}} \sum_{k=0}^{\infty} F_k \int_{C_+} \frac{(-ns)^{-(\mu-k)} e^{-ns} ds}{2\pi i\, (-n)^{k-\mu}}$$

$$= \frac{n!\,2^{\mu-1}B^n}{(\pi i)^{\mu+n+1}e^{(\mu-D)\pi i}} \sum_{k=0}^{\infty} F_k\; H_k,$$

where

$$H_k = \frac{1}{(-n)^{k-\mu}} \frac{1}{2\pi i} \int_{C_+} e^{-ns}(-ns)^{-(\mu-k)}\, ds. \tag{15}$$

Now, evaluate H_k. Let $t = ns$. Then $dt = nds, s = \frac{t}{n}, ds = \frac{dt}{n}$,

$$H_k = -\frac{1}{(-n)^{k-\mu+1}} \frac{1}{2\pi i} \int_{C_+} e^{-t}(-t)^{-(\mu-k)}\, dt.$$

Be deformation of paths, and using the reciprocal Gamma function,

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_H e^{-t}\, (-t)^{-z}\, dt,$$

where H is the Hankel contour, we have

$$H_k = \frac{1}{(-n)^{k-\mu+1}} \frac{i}{2\pi} \int_H e^{-t}(-t)^{-(\mu-k)}\; dt = (-1)^{\mu-k-1}\, (n)^{\mu-k-1} \frac{1}{\Gamma(\mu-k)}.$$

RECOLETOS MULTIDISCIPLINARY RESEARCH JOURNAL

Moreover,

$$H_k = (-1)^{\mu-1}(-1)^{-k}\, n^{\mu-k-1}\frac{1}{\Gamma(\mu-k)} = (-1)^{\mu-1}(-1)^{k}\, n^{\mu-k-1}\frac{1}{\Gamma(\mu-k)}$$
$$= e^{\pi i(\mu-1)}\, n^{\mu-k-1}\,\frac{(-1)^k}{\Gamma(\mu-k)}.$$

Since $\Gamma(\mu-k) = (\mu-k-1)!$, and

$$\langle x\rangle_k = x(x+1)(x+2)\cdots(x+k-1)$$
$$\langle 1-\mu\rangle_k = (1-\mu)(2-\mu)(3-\mu)\cdots(k-\mu),$$

then

$$\frac{(-1)^k}{\Gamma(\mu-k)} = \frac{(-1)^k}{(\mu-k-1)!} = \frac{(-1)^k\,(\mu-1)(\mu-2)(\mu-3)\cdots(\mu-k)}{(\mu-1)!} = \frac{\langle 1-\mu\rangle_k}{\Gamma(\mu)}.$$

Thus,

$$H_k = n^{\mu-k-1}\, e^{\pi i(\mu-1)}\,\frac{\langle 1-\mu\rangle_k}{\Gamma(\mu)}.$$

Writing $i^{n+\mu} = e^{(n+\mu)\frac{\pi i}{2}}$,

$$I_+ = \frac{n!\,2^{\mu}B^{n}}{(\pi i)^{\mu+n}\, e^{(\mu-D)\pi i}}\sum_{k=0}^{\infty} F_k H_k \quad = \frac{n!\,2^{\mu}B^{n}}{(\pi i)^{\mu+n}\, e^{(\mu-D)\pi i}}\sum_{k=0}^{\infty} F_k n^{\mu-k-1}\, e^{\pi i(\mu-1)}\frac{\langle 1-\mu\rangle_k}{\Gamma(\mu)}$$
$$= \frac{n!\,2^{\mu}B^{n}\, n^{\mu-1}\, e^{(2D-\mu-n-2)\pi i/2}}{\pi^{\mu+n}\,\Gamma(\mu)}\sum_{k=0}^{\infty} F_k\frac{\langle 1-\mu\rangle_k}{\Gamma(\mu)}.$$

Since $e^{(2D-\mu-n-2)\pi i/2} = \cos[(2D-\mu-n-2)\pi/2] + i\sin[(2D-\mu-n-2)\pi/2]$, then

$$I_+ = \frac{n!\,2^{\mu}B^{n}\, n^{\mu-1}}{\pi^{\mu+n}\,\Gamma(\mu)}(\cos\beta + i\sin\beta)\left[\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k}\, Re(F_k) + i\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k} Im(F_k)\right]$$
$$= \frac{n!\,2^{\mu}B^{n}\, n^{\mu-1}}{\pi^{\mu+n}\,\Gamma(\mu)}\left[\cos\beta\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k}\, Re(F_k) + i\sin\beta\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k} Re(F_k)\right.$$
$$\left. + i\cos\beta\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k}\, Im(F_k) - \sin\beta\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k} Im(F_k)\right],$$

where $\beta = (2D-\mu-n-2)\frac{\pi}{2}$.

The integral along L_- denoted by I_- can be obtained similarly, with $t = -\frac{\pi i e^s}{B}$. It can B be shown that I_- is the complex conjugate of I_+ (not considering x and μ as complex numbers). Thus,

$$T_n^{(\mu)}(x;a,b,c) = I_+ + I_- = 2\,Re(I_+).$$

Hence,

$$T_n^{(\mu)}(x;a,b,c) = \frac{n!\,2^{\mu+1}B^{n}n^{\mu-1}}{\pi^{\mu+n}\,\Gamma(\mu)}\left[\cos\beta\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k}\, Re(F_k) - \sin\beta\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k}\, Im(F_k)\right].$$

By setting $b = c = e$ and $a = 1$ in Theorem 2.2, we derive the following corollary. ∎

Corollary 2.3. [Corcino and Corcino (2022), Theorem 3] *Under the conditions of Theorem 2.2 as* $n \to +\infty$,

$$T_n^{(\mu)}(x;1,e,e) = T_n^{\mu}(x) \sim \frac{n!\,2^{\mu+n+1}n^{\mu-1}}{\pi^{\mu+n}\,\Gamma(\mu)}\left[\cos\beta\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k}Re(F_k) - \sin\beta\sum_{k=0}^{\infty}\frac{\langle 1-\mu\rangle_k}{n^k}Im(F_k)\right]$$

where $\beta = (x-\mu-n-2)\frac{\pi}{2}$.

Computing the first few values of $F_k^{(r)}$ and $F_k^{(i)}$ using Mathematica yields:

$$\begin{aligned}
F_0^{(r)} &= 1,\\
F_0^{(i)} &= 0,\\
F_1^{(r)} &= -\frac{\mu}{2},\\
F_1^{(i)} &= \frac{n\pi}{2} + \beta,\\
F_2^{(r)} &= \frac{1}{24}(-12\beta^2 - 12\beta n\pi - 3n^2\pi^2 - \mu + \pi^2\mu + 3\mu^2),\\
F_2^{(i)} &= -\frac{1}{4}(2\beta + n\pi)(-1+\mu),\\
F_2^{(r)} &= \frac{1}{48}(12\beta^2(-2+\mu) + 12\beta n\pi(-2+\mu) + 3n^2\pi^2(-2+\mu)\\
&\quad + \mu(-\pi^2(-2+\mu) - (-1+\mu)\mu)),\\
F_3^{(i)} &= -\frac{1}{48}(2\beta + n\pi)(-4 + 4\beta^2 + 4\beta n\pi + n^2\pi^2 + 7\mu - \pi^2 - 3\mu^2).
\end{aligned}$$

The first-order approximation is given by the following theorem.

Theorem 2.4. *As* $n \to +\infty$, μ, x *are fixed complex numbers, and* $\beta = (2D-\mu-n-2)\frac{\pi}{2}$ where $D = \frac{x\ln c - \mu\ln a}{B}$ and $B = 2\ln b - \ln a$,

$$T_n^{(\mu)}(x;a,b,c) \sim \frac{n!\,2^{\mu+1}B^n n^{\mu-1}}{\pi^{\mu+n}\Gamma(\mu)}\left\{\cos\beta + O\left(\frac{1}{n}\right)\right\}.$$

Proof. By taking F_0 for F_k and taking the first term of the sum, where $F_0 = F(0) = 1$,

$$\begin{aligned}
T_n^{(\mu)}(x;a,b,c) &\sim \frac{n!\,2^{\mu+1}B^n n^{\mu-1}}{\pi^{\mu+n}\Gamma(\mu)}\{\cos\beta\,(Re(F_0) - \sin\beta\,(Im(F_0))\}.\\
&\sim \frac{n!\,2^{\mu+1}B^n n^{\mu-1}}{\pi^{\mu+n}\Gamma(\mu)}\{\cos\beta\,(1) - \sin\beta\,(0)\}.\\
&\sim \frac{n!\,2^{\mu+1}B^n n^{\mu-1}}{\pi^{\mu+n}\Gamma(\mu)}\left\{\cos\beta + O\left(\frac{1}{n}\right)\right\}.
\end{aligned}$$

∎

Corollary 2.5. [Corcino and Corcino (2022), Theorem 4] *Under the conditions of Theorem 2.4 as* $n \to +\infty$

$$T_n^{(\mu)}(x;1,e,e) = T_n^{(\mu)}(x) \sim \frac{n!2^{\mu+n+1}n^{\mu-1}}{\pi^{\mu+n}\Gamma(\mu)}\left\{\cos\beta + O\left(\frac{1}{n}\right)\right\} \qquad (16)$$

where $\beta = (x-\mu-n-2)\frac{\pi}{2}$.

Proof. Take $b = c = e$ and $a = 1$ in Theorem 2.4.

Asymptotic expansions of Apostol-tangent polynomials of complex order μ with parameters $\boldsymbol{a, b, c}$

We apply the same method as the previous section. For convenience, we take $\lambda = e^{2\xi}$ where
$\xi = \frac{\log \lambda}{2}$ and $|\xi| < \frac{\pi}{2}$. Computation for the singularities of the generating function in (2) is done as follows:

$$\begin{aligned} \lambda e^{2t\ln b} + e^{t\ln a} &= 0 \\ e^{2\xi + 2t\ln b - t\ln a} &= -1 \\ 2\xi + 2t\ln b - t\ln a &= (2k+1)\pi i \\ t_k := t &= \frac{(2k+1)\pi i - 2\xi}{B}, k \in \mathbb{Z} \end{aligned}$$

With fixed λ, these singularities lie on a vertical line to the left or the right of the origin depending on whether ν, the real part of ξ, is positive or negative (See Figure 2).
The singularities nearest to the origin are t_0^* and t_{-1}^*, which are given by

$$t_0^* = \frac{\pi i - 2\xi}{B}, \quad t_{-1}^* = \frac{\pi i - 2\xi}{B} \tag{17}$$

Applying Cauchy's integral formula to (2), we have

$$T_n^{(\mu)}(x; \lambda, a, b, c) = \frac{n!}{2\pi i} \int_c \left(\frac{2}{e^{2t\ln b + 2\xi} + e^{t\ln a}} \right) e^{xt\ln c} \frac{dt}{t^{n+1}} \tag{18}$$

where C is a circle about zero with radius $< min\{|t_0^*|, |t_{-1}^*|\}$. These singularities are the sources of the main asymptotic contribution. We integrate around a circle C_2 about zero with radius $= min\{|t_0^*|, |t_{-1}^*|\} + \epsilon$. The choice of the radius of C_2 is such that only the singularities 0, t_0^* andt_{-1}^*. Moreover, we integrate around C_2 avoiding the branch cuts from t_0^* to $+\infty$ and t_{-1}^* to $+\infty$. Refer again to Figure 2.

Denote the loops by L_+^* and L_-^* and the remaining part of the circle C_2 by C^{**}. Then

$$\frac{n!}{2\pi i} \int_{C_2} g(t)\, dt = \frac{n!}{2\pi i} \left[\int_{C^{**}} g(t)\, dt + \int_{L_+^*} g(t)\, dt + \int_{L_-^*} g(t)\, dt \right] \tag{19}$$

where $g(t)$ is the integrand on the right-hand side of (18). By the principle of deformation of paths,

$$\frac{n!}{2\pi i} \int_{C_2} g(t)\, dt = \frac{n!}{2\pi i} \int_C g(t)\, dt = T_n^{(\mu)}(x; \lambda, a, b, c) \tag{20}$$

and

$$T_n^{(\mu)}(x; \lambda, a, b, c) = \frac{n!}{2\pi i} \left[\int_{C^{**}} g(t)\, dt + \int_{L_+^*} g(t)\, dt + \int_{L_-^*} g(t)\, dt \right]. \tag{21}$$

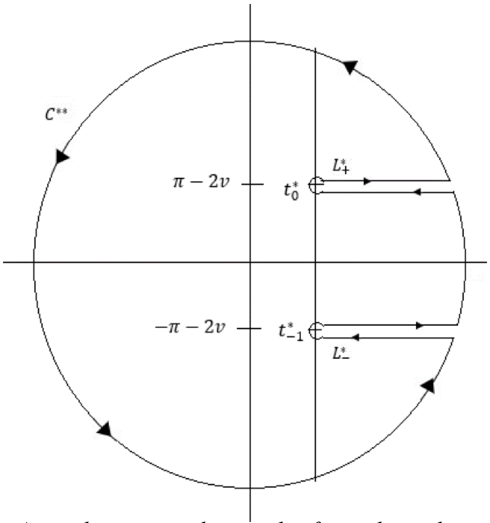

Figure 2. *Contour for Apostol-tangent polynomials of complex order μ with parameters a,b and c*

Corollary 3.1. *It follows from Lemma 2.1 that the contribution from the circular arc C^{**} is also $O((\pi)^{-n})$, so that, for large values of n (as n → +∞), it is exponentially small with respect to the main contribution.*

Theorem 3.2. *As $n \to +\infty$, μ, x and λ are fixed complex numbers and* $\rho = \frac{\log \lambda}{2\pi i}$,

$$T_n^{(\mu)}(x;\lambda,a,b,c) \sim \frac{n!\,2^{\mu}B^n n^{\mu-1}e^{-2\rho D\pi i}}{\pi^{\mu+n}\Gamma(\mu)}\left[\frac{e^{iB}}{(1-2\rho)^{\mu+n}}\sum_{k=0}^{\infty} h_k \frac{\langle 1-\mu\rangle_k}{n^k} + \frac{e^{-iB}}{(1+2\rho)^{\mu+n}}\sum_{k=0}^{\infty} f_k \frac{\langle 1-\mu\rangle_k}{n^k} + \right]$$

Proof. We compute the contributions from the loops L_+^* and L_-^*. Let I_+^* be the integral along the loop L_+^*. For computation purposes, write $\lambda = e^{2\rho\pi i}$. Then

$$I_+^* = \frac{n!}{2\pi i}\int_{L_+^*} \frac{2^{\mu}e^{xt\ln c}}{(e^{2t\ln b+2\rho\pi i} + e^{t\ln a})^{\mu}}\frac{dt}{t^{n+1}} \tag{22}$$

Let $t = \left(\frac{1-2\rho}{B}\right)\pi i e^s$. Then $dt = \left(\frac{1-2\rho}{B}\right)\pi i e^s ds$ and

$$I_+^* = \frac{n!}{2\pi i}\int_{C_+^*} \frac{2^{\mu}e^{x\left(\frac{1-2\rho}{B}\right)\pi i e^s \ln c}}{\left(e^{2\left(\frac{1-2\rho}{B}\right)\pi i e^s \ln b+2\rho\pi i} + e^{\left(\frac{1-2\rho}{B}\right)\pi i e^s \ln a}\right)^{\mu}} \frac{\left(\frac{1-2\rho}{B}\right)\pi i e^s d}{\left(\left(\frac{1-2\rho}{B}\right)\pi i e^s\right)^{n+1}}$$

$$= \frac{n!2^{\mu-1}B^n}{(\pi i)^{n+1}(1-2\rho)^n}\int_{C_+^*} \frac{e^{\left(\frac{1-2\rho}{B}\right)\pi i e^s x\ln c}ds}{\left(e^{\left(\frac{1-2\rho}{B}\right)\pi i e^s \ln a}\right)^{\mu}\left(e^{2\left(\frac{1-2\rho}{B}\right)\pi i e^s \ln b-\left(\frac{1-2\rho}{B}\right)\pi i e^s \ln a+2\rho\pi i}+1\right)^{\mu} e^{sn}}$$

$$= \frac{n!\,2^{\mu-1}B^n}{(\pi i)^{n+1}(1-2\rho)^n e^{\pi i u}}\int_{C_+^*} \frac{e^{\left(\frac{1-2\rho}{B}\right)\pi i e^s (x\ln c-\mu\ln a)}ds}{(e^{(1-2\rho)\pi i e^s+2\rho\pi i-\pi i} - 1)^{\mu}e^{sn}}.$$

RECOLETOS MULTIDISCIPLINARY RESEARCH JOURNAL

Multiplying $e^{\frac{\pi i(1-2\rho)}{B}(x\ln c-\mu\ln a)}e^{-\frac{\pi i(1-2\rho)}{B}(x\ln c-\mu\ln a)}$ and $\left[\left(\frac{1-2\rho}{B}\right)\pi i\right]^{\mu}\left[\left(\frac{1-2\rho}{B}\right)\pi i\right]^{-\mu}$, then

$$I_+^* = \frac{n!\,2^{\mu-1}B^n}{(\pi i)^{n+1}(1-2\rho)^n e^{\pi i u}}\int_{C_+^*}\frac{e^{\left(\frac{1-2\rho}{B}\right)\pi i e^s(x\ln c-\mu\ln a)}ds}{(e^{(1-2\rho)\pi i e^s+2\rho\pi i-\pi i}-1)^{\mu}e^{sn}}$$
$$\times e^{\frac{\pi i(1-2\rho)}{B}(x\ln c-\mu\ln a)}e^{-\frac{\pi i(1-2\rho)}{B}(x\ln c-\mu\ln a)}\left[\left(\frac{1-2\rho}{B}\right)\pi i\right]^{\mu}\left[\left(\frac{1-2\rho}{B}\right)\pi i\right]^{-\mu}$$
$$= \frac{n!\,2^{\mu-1}B^{\mu+n}}{(\pi i)^{\mu+n+1}(1-2\rho)^{\mu+n}e^{[\mu-(1-2\rho)D]\pi i}}\int_{C_+^*}\left[\left(\frac{1-2\rho}{B}\right)\pi i\right]^{\mu}\frac{e^{Dv*}ds}{(e^{v*}-1)^{\mu}e^{sn}}$$
$$= \frac{n!\,2^{\mu-1}B^{n}}{(\pi i)^{\mu+n+1}(1-2\rho)^{\mu+n}e^{[\mu-(1-2\rho)D]\pi i}}\int_{C_+^*}\left[\left(\frac{(1-2\rho)\pi i}{e^{v*}-1}\right)\right]^{\mu}e^{Dv*}e^{-sn}ds,$$

where $v* = (1-2\rho)(e^s-1)\pi i$ and $D = \frac{x\ln c-\mu\ln a}{B}$. Multiplying the last array by $s^{\mu}s^{-\mu}$, then

$$I_+^* = \frac{n!\,2^{\mu-1}B^{n}}{(\pi i)^{\mu+n+1}(1-2\rho)^{\mu+n}e^{[\mu-(1-2\rho)D]\pi i}}\int_{C_+^*}\left[\left(\frac{(1-2\rho)\pi is}{e^{v*}-1}\right)\right]^{\mu}e^{Dv*}e^{-sn}s^{-\mu}ds,$$
$$= \frac{n!\,2^{\mu-1}B^{n}}{(\pi i)^{\mu+n+1}(1-2\rho)^{\mu+n}e^{[\mu-(1-2\rho)D]\pi i}}\int_{C_+^*}h(s)e^{-sn}s^{-\mu}ds, \tag{23}$$

where

$$h(s) = \left[\left(\frac{(1-2\rho)\pi is}{e^{v*}-1}\right)\right]^{\mu}e^{Dv*}.$$

By expanding $h(s) = \sum_{k=0}^{\infty}h_k s^k$, (23) becomes

$$I_+^* = \frac{n!\,2^{\mu-1}B^{n}}{(\pi i)^{\mu+n+1}(1-2\rho)^{\mu+n}e^{[\mu-(1-2\rho)D]\pi i}}\int_{C_+^*}\sum_{k=0}^{\infty}h_k s^k\,e^{-sn}s^{-\mu}ds,$$
$$= \frac{n!\,2^{\mu}B^{n}}{(\pi i)^{\mu+n}(1-2\rho)^{\mu+n}e^{[\mu-(1-2\rho)D]\pi i}}\sum_{k=0}^{\infty}h_k H_k,$$

where

$$H_k = \frac{1}{(-n)^{k-\mu}}\frac{1}{2\pi i}\int_{C_+}e^{-ns}(-ns)^{k-\mu}\,ds$$
$$= n^{\mu-k-1}\,e^{\pi i\mu}\frac{\langle 1-\mu\rangle_k}{\Gamma(\mu)}.$$

So,

$$I_+^* = \frac{n!\,2^{\mu}B^{n}}{(\pi i)^{\mu+n}(1-2\rho)^{\mu+n}e^{[\mu-(1-2\rho)D]\pi i}}\sum_{k=0}^{\infty}h_k n^{\mu-k-1}\,e^{\pi i\mu}\frac{\langle 1-\mu\rangle_k}{\Gamma(\mu)},$$
$$= \frac{n!\,2^{\mu}B^{n}e^{-2\rho D\pi i}\,e^{[2D-\mu-n]\frac{\pi i}{2}}n^{\mu-1}}{(\pi)^{\mu+n}(1-2\rho)^{\mu+n}\Gamma(\mu)}\sum_{k=0}^{\infty}h_k\frac{\langle 1-\mu\rangle_k}{\mathrm{n}^{\mathrm{k}}},$$
$$= \frac{n!\,2^{\mu}B^{n}e^{-2\rho D\pi i}\,e^{i\beta}n^{\mu-1}}{(\pi(1-2\rho))^{\mu+n}\Gamma(\mu)}\sum_{k=0}^{\infty}h_k\frac{\langle 1-\mu\rangle_k}{\mathrm{n}^{\mathrm{k}}}.$$

Next, let I_{-}^{*} be the integral along the loop L_{-}^{*}. Then,

$$I_{-}^{*}=\frac{n!}{2\pi i}\int_{L_{-}^{*}}\frac{2^{\mu}e^{xt\ln c}}{(e^{2t\ln b+2\rho\pi i}+e^{t\ln a})^{\mu}}\frac{dt}{t^{n+1}}. \tag{24}$$

After the substitution $t=\left(\frac{1-2\rho}{B}\right)\pi ie^{s}$, we obtain

$$\begin{aligned}I_{-}^{*}&=\frac{n!\,2^{\mu-1}B^{n}}{(\pi i)^{\mu+n+1}(-1-2\rho)^{\mu+n}e^{[\mu-(-1-2\rho)D]\pi i}}\int_{C_{-}^{*}}\left[\left(\frac{(-1-2\rho)\pi i}{e^{v*}-1}\right)\right]^{\mu}e^{Dv*}e^{-sn}ds,\\&=\frac{n!\,2^{\mu-1}B^{n}}{(\pi i)^{\mu+n+1}(-1-2\rho)^{\mu+n}e^{[\mu-(-1-2\rho)D]\pi i}}\int_{C_{-}^{*}}f(s)s^{\mu}e^{-sn}ds,\end{aligned}$$

where

$$f(s)=\left[\left(\frac{(-1-2\rho)\pi is}{e^{v*}-1}\right)\right]^{\mu}e^{Dv*}.$$

Expand $f(s)=\sum_{k=0}^{\infty}f_{k}s^{k}$ and interchange the summation and integration. Note that $(-i)^{\mu+n}=e^{-(\mu+n)\frac{\pi i}{2}}$. Thus,

$$I_{-}^{*}=\frac{n!\,2^{\mu}B^{n}e^{-2\rho D\pi i}\,e^{iB}n^{\mu-1}}{(\pi\,(1+2\rho))^{\mu+n}\Gamma(\mu)}\sum_{k=0}^{\infty}f_{k}\frac{\langle 1-\mu\rangle_{k}}{\mathrm{n}^{\mathrm{k}}},$$

where $-\beta=(\mu+n-2D)\frac{\pi}{2}$. Combining I_{+}^{*} and I_{-}^{*} will give,

$$\begin{aligned}T_{n}^{(\mu)}(x;\lambda,a,b,c)&=\ I_{+}^{*}+\ I_{-}^{*}\\&=\frac{n!\,2^{\mu}B^{n}e^{-2\rho D\pi i}\,e^{i\beta}n^{\mu-1}}{(\pi(1-2\rho))^{\mu+n}\Gamma(\mu)}\sum_{k=0}^{\infty}h_{k}\frac{\langle 1-\mu\rangle_{k}}{\mathrm{n}^{\mathrm{k}}}+\frac{n!\,2^{\mu}B^{n}e^{-2\rho D\pi i}\,e^{iB}n^{\mu-1}}{(\pi\,(1+2\rho))^{\mu+n}\Gamma(\mu)}\sum_{k=0}^{\infty}f_{k}\frac{\langle 1-\mu\rangle_{k}}{\mathrm{n}^{\mathrm{k}}}\\&=\frac{n!\,2^{\mu}B^{n}n^{\mu-1}e^{-2\rho D\pi i}}{\pi^{\mu+n}\Gamma(\mu)}\left[\frac{e^{i\beta}}{(1-2\rho)^{\mu+n}}\sum_{k=0}^{\infty}h_{k}\frac{\langle 1-\mu\rangle_{k}}{\mathrm{n}^{\mathrm{k}}}+\frac{e^{-i\beta}}{(1+2\rho)^{\mu+n}}\sum_{k=0}^{\infty}f_{k}\frac{\langle 1-\mu\rangle_{k}}{\mathrm{n}^{\mathrm{k}}}\right].\end{aligned}$$

For $b=c=e$ and $a=1$, we have the following theorem.

Corollary 3.3. [Corcino et al. (2023), Theorem 7] *Under the conditions of Theorem 3.2 as $n\to+\infty$,*

$$\begin{aligned}T_{n}^{(\mu)}(x;\lambda,1,e,e)\sim\frac{n!\,2^{\mu+n}n^{\mu-1}e^{-2\rho\pi i}}{\pi^{\mu+n}\Gamma(\mu)}&\left[\frac{e^{i\beta}}{(1-2\rho)^{\mu+n}}\sum_{k=0}^{\infty}h_{k}\frac{\langle 1-\mu\rangle_{k}}{n^{k}}\right.\\&\left.+\frac{e^{-i\beta}}{(1+2\rho)^{\mu+n}}\sum_{k=0}^{\infty}f_{k}\frac{\langle 1-\mu\rangle_{k}}{n^{k}}+\right]\end{aligned}$$

where $\beta=(x-\mu-n)\frac{\pi}{2}$.

Corollary 3.4. *When $\rho=0$, Theorem 3.2 reduces to Theorem 2.2.*

Computing the first few values of h_k and f_k using Mathematica yields:

$$h_0^{(r)} = 1,$$
$$h_0^{(i)} = 0,$$
$$h_1^{(r)} = -\frac{\mu}{2},$$
$$h_1^{(i)} = -\frac{1}{2}(1-2\rho) + (2\beta + n\pi),$$
$$h_2^{(r)} = \frac{1}{24}(-12\beta^2(1-2\rho)^2 - 12\beta(1-2\rho)^2 n\pi - 3(1-2\rho)^2 n^2\pi^2 + \mu(-1 + (1-2\rho)^2\pi^2 + 3\mu\,)\,,$$
$$h_2^{(i)} = \frac{1}{4}(-12\rho)(2\beta + n\pi)(-1+\mu),$$

and

$$f_0^{(r)} = 1,$$
$$f_0^{(i)} = 0,$$
$$f_1^{(r)} = -\frac{\mu}{2},$$
$$f_1^{(i)} = -\frac{1}{2}(1+2\rho) + (2\beta + n\pi),$$
$$f_2^{(r)} = \frac{1}{24}(-12(\beta + 2\beta\rho)^2 - 12\beta(1+2\rho)^2 n\pi - 3(\pi + 2\rho\pi)^2 n^2 + \mu(-1+(1+2\rho)^2\pi^2 + 3\mu\,)\,,$$
$$f_2^{(i)} = \frac{1}{4}(-12\rho)(2\beta + n\pi)(-1+\mu),$$

A first-order approximation is obtained by taking h_0 and f_0 for h_k and f_k, respectively, and taking the first term of the sum. This is given in the following theorem.

Theorem 3.5. *As* $n \to +\infty$, μ *and* x *are fixed complex numbers,*

$$T_n^{(\mu)}(x;\lambda,a,b,c) \sim \frac{n!\,2^{\mu}B^{n}n^{\mu-1}}{\pi^{\mu+n}\Gamma(\mu)}\left\{\frac{e^{i\beta}(1+2\rho)+e^{-i\beta}(1-2\rho)}{(1-4\rho^2)^{\mu+n}} + O\left(\frac{1}{n}\right)\right\}.$$

where $\beta = (x-\mu-n)\frac{\pi}{2}$.

Take $\rho = 0, b = c = e,$ and $a = 1$, we have the following corollary.

Corollary 3.6. [Corcino et al. (2023), Theorem 9] *Under the conditions of Theorem 3.5 as* $n \to +\infty$,

$$T_n^{(\mu)}(x;1,1,e,e,) \sim \frac{n!\,2^{\mu+n}n^{\mu-1}}{\pi^{\mu+n}\Gamma(\mu)}\left\{2\cos\beta + O\left(\frac{1}{n}\right)\right\}$$

where $\beta = (x-\mu-n)\frac{\pi}{2}$.

Remark 3.7. When ρ = 0, Theorem 3.5 reduces to Theorem 2.4.

CONCLUSION

In conclusion, this paper has endeavored to extend the exploration of asymptotic approximations, explicitly focusing on tangent and Apostol-tangent polynomials of complex order μ with parameters a, b, and c. Building upon methodologies employed in prior studies, such as Corcino et al. (2023) and López and Temme (2010), which successfully obtained asymptotic approximations for various polynomials, our investigation seeks to apply similar techniques to unveil insightful results regarding the tangent and Apostol-tangent polynomials. The foundation laid by previous research, particularly in the study of Bernoulli, Euler, and Genocchi polynomials and the innovative approaches demonstrated in works like Yasmin and Muhyi (2021), provides a robust framework for our inquiry. Given the historical significance and widespread applications of tangent polynomials in mathematics and physics, pursuing asymptotic formulas for this new class of polynomials promises to contribute valuable insights to the mathematical community.

Recommendations for future research involve exploring the broader implications and applications of the derived asymptotic formulas for tangent and Apostol-tangent polynomials. Additionally, further investigations into the properties and behaviors of the newly introduced 2-variable q-generalized tangent-Apostol-type polynomials, as studied in Yasmin and Muhyi (2021), could pave the way for deeper understanding and potential applications in various scientific disciplines. Collaborative efforts across mathematical and scientific communities may explore these polynomials more comprehensively, unlocking new avenues for theoretical advancements and practical implementations.

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