Perfect Outer-connected Domination in the Join and Corona of Graphs

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Abstract

Let G be a connected simple graph. A dominating set $S \subseteq V(G)$ is called a *perfect dominating set* of G if each $u \in V(G) \setminus S$ is dominated by exactly one element of S. A set S of vertices of a graph G is an outer-connected dominating set if every vertex not in S is adjacent to some vertex in S and the subgraph induced by $V(G) \setminus S$ is connected. A perfect dominating set S of a graph G is a perfect outer-connected dominating set if the subgraph induced by $V(G) \setminus S$ is connected. The perfect outer-connected domination number of G, denoted by $\tilde{\gamma}_c^p(G)$, is the smallest cardinality of a perfect outer-connected dominating set S of G. A perfect outer-connected dominating set with cardinality $\tilde{\gamma}_c^p(G)$ is called $\tilde{\gamma}_c^p$-*set* of G. In this paper, we will show that given positive integers, a, b, c, and n such that $a \leq b \leq c \leq n-1$, there exists a connected graph G with $|V(G)| = n$, $\gamma(G) = a$, $\gamma_p(G) = b$, and $\tilde{\gamma}_c^p(G) = c$. Further, we give the characterization of the perfect outer-connected dominating set of the join and corona of two graphs and give their corresponding perfect outer-connected domination number.

Keywords: perfect dominating set, outer-connected dominating set, perfect outer-connected dominating set.

1. Introduction

Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962. One type of domination in graphs is the perfect domination. This was introduced by Cockayne et.al. The inverse perfect domination in graphs was introduced by Salve and Enriquez. The outer connected domination in graph is found in the paper of Cyman and Raczek and in the paper of Chang, Liu, and Wang. Computing a minimum outer-connected dominating set can be read in the paper of Mark Keil and Pradhan. In this paper, we introduce the concept of perfect outer-connected domination in graphs and apply this concept using the binary operations such as the join and corona of two graphs. For general concepts we refer the reader to.

Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G): uv \in E(G)\}$. If $S \subseteq V(G)$, then the *open neighborhood* of S is the set $N_G(S) = N(S) = \cup_{v \in S} N_G(v)$. The *closed neighborhood* of S is $N_G[S] = N[S] = S \cup N(S)$. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in (V(G) \setminus S)$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G.

A dominating set $S \subseteq V(G)$ is called a *perfect dominating set* of G if each $u \in V(G) \setminus S$ is dominated by exactly one element of S. The *perfect domination number* of G, denoted by $\gamma_p(G)$, is

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the minimum cardinality of a perfect dominating set of G. A perfect dominating set of cardinality $\gamma_p(G)$ is called a γ_p-set of G. A set S of vertices of a graph G is an outer-connected dominating set if every vertex not in S is adjacent to some vertex in S and the subgraph induced by $V(G) \setminus S$ is connected. The outer-connected domination number $\tilde{\gamma}_c(G)$ is the minimum cardinality of the outer-connected dominating set S of a graph G. An outer-connected dominating set of cardinality $\tilde{\gamma}_c(G)$ is called a $\tilde{\gamma}_c$-set of G. A perfect dominating set S of a graph G is a perfect outer-connected dominating set if the subgraph induced by $V(G) \setminus S$ is connected. The perfect outer-connected domination number of G, denoted by $\tilde{\gamma}_c^p(G)$, is the smallest cardinality of a perfect outer-connected dominating set S of G. A perfect outer-connected dominating set with cardinality $\tilde{\gamma}_c^p(G)$ is called $\tilde{\gamma}_c^p$-set of G.

As an application, we consider a computer network in which a core group of fileservers has the ability to communicate directly with every computer outside the core group. In addition, every computer outside the core group is connected exactly with one fileserver. Moreover, every two computers outside the core group may directly communicate with each other without intervention of any of the fileservers to protect the fileservers from overloading, or every two computers outside the core group may be connected through another computer outside the core group. A smallest core group with these properties is a $\tilde{\gamma}_c^p$-set for the graph representing the network.

2. Results

From the definitions, the following result is immediate.

Remark 2.1: Let G be a connected graph of order $n \geq 2$. Then $\gamma(G) \leq \gamma_p(G) \leq \tilde{\gamma}_c^p(G)$.

Theorem 2.2: Let a, b, c, and $n \geq 2$ be positive integers such that $a \leq b \leq c \leq n-1$. Then there exists a connected graph G with $|V(G)| = n$ such that $\gamma(G) = a$, $\gamma_p(G) = b$, and $\tilde{\gamma}_c^p(G) = c$.

Proof: Consider the following cases:

*Case*1. Suppose $a = b = c < n-1$.

Let $G = P_a \circ K_1$ (see Figure 1) and let $n = 2a$.

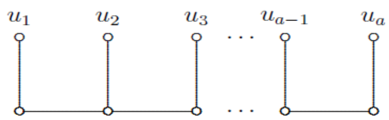

Figure 1: A graph G with $a = b = c < n-1$

The set $A = \{u_i : i = 1,2,\ldots,a\}$ is a γ-set, γ_p-set, and $\tilde{\gamma}_c^p$-set of G. Thus, $|V(G)| = 2|A| = 2a = n$, $\gamma(G) = |A| = a$, $\gamma_p(G) = |A| = b$, and $\tilde{\gamma}_c^p(G) = |A| = c$.

*Case*2. Suppose $a = b < c < n-1$.

Let $a = k+1$ for some $k \in \setminus N, n = 3a$ and $c = 2a$. Consider the graph G obtained from the graph in Figure 1 by adding the vertex x_i and the edges u_ix_i for $i = 1,2,\ldots,a$ (see Figure 2).

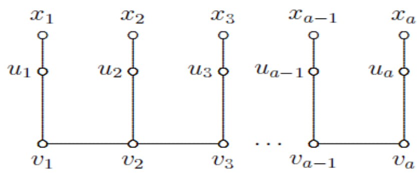

Figure 2: A graph G with $a = b < c < n-1$.

The set $A = \{u_i : i = 1,2,\ldots,a\}$ is a γ-*set* and γ_p-*set* of G, and $B = A \cup \{x_i : i = 1,2,\ldots,a\}$ is a $\tilde{\gamma}_c^p$-*set* Thus, $|V(G)| = 3|A| = 3a = n$, $\gamma(G) = |A| = a$, $\gamma_p(G) = |A| = a = b$, and $\tilde{\gamma}_c^p(G) = |B| = 2a = c$.

*Case*3. Suppose $a < b = c < n-1$.

Let $a = 2k+1$ for some $k \in \mathbb{N}$, $2b = 3a-1$, and $n = 3a-1$. Consider the graph G obtained from $P_a = [x_1, x_2, \ldots, x_a]$ and $P_{2a-1} = [v_1, w_1, \ldots, v_{a-1}, w_{a-1}, v_a]$ by adding the edges $x_i v_i$ for all $i = 1,2,\ldots,a$ (see Figure 3).

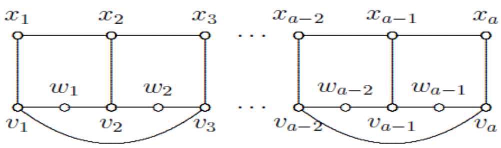

Figure 3: A graph G with $a < b = c < n-1$.

The sets $A = \{v_i : i = 1,2,\ldots,a\}$ is a γ-*set* of G, $B = V(P_a) \cup \{v_{2i} : i = 1,2,\ldots,\frac{a-1}{2}\}$ is a γ_p-*set* and $\tilde{\gamma}_c^p$-*set*. Thus, $|V(G)| = |V(P_a)| + |V(P_{2a-1})| = 3a-1 = n$, $\gamma(G) = |A| = a$, $\gamma_p(G) = |B| = a + (a-1)/2 = (3a-1)/2 = b$, and $\tilde{\gamma}_c^p(G) = |B| = b = c$.

*Case*4. $a < b < c < n-1$.

Let $a = 2k+1$ for some $k \in \mathbb{N}$, $2b = 3a-1$, and $2c = 2n - 3a + 1$. Consider the graph G obtained from the graph in Figure 3 by adding the vertices $y_1, y_2, \ldots y_{n-3a+1}$ the edges $x_a y_i$ for all $i = 1,2,\ldots,n-3a+1$ (see Figure 4).

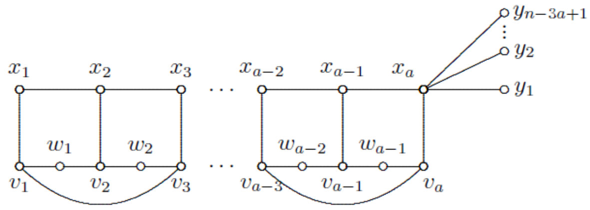

Figure 4: A graph G with $a < b < c < n - 1$.

The sets $A = \{ v_i : i = 1,2,\ldots, a-1 \} \cup \{ x_a \}$ is a γ-set of G, $B = V(P_a) \cup \{ v_{2i} : i = 1,2,\ldots,\frac{a-1}{2} \}$ is a γ_p-set and $C = B \cup \{ y_i : i = 1,2,\ldots, n-3a+1 \}$ is a $\tilde{\gamma}_c^p$-set. Thus, $|V(G)| = |V(P_a)| + |V(P_{2a-1})| + |\{ y_i : i = 1,2,\ldots, n-3a+1\}| = a + (2a-1) + (n-3a+1) = n$, $\gamma(G) = |A| = (a-1) + 1 = a, \gamma_p(G) = |B| = (a) + \frac{a-1}{2} = \frac{3a-1}{2} = b$, and $\tilde{\gamma}_c^p(G) = |C| = b + (n-3a+1) = \frac{3a-1}{2} + (n - 3a + 1) = \frac{2n-3a+1}{2} = c$.

*Case*5. Suppose $a < b < c = n - 1$.

Consider the graph G obtained from $P_{2a-1} = [v_1, w_1, \ldots, v_{a-1}, w_{a-1}, v_a]$ by adding the vertices x_i, y_i and the edges $v_i x_i, v_i y_i$ for all $i = 1,2,\ldots, a$ and adding the vertices u_j and the edges $v_a u_j$ for all $j = 1,2,\ldots, c-4a+2$ (see Figure 5). Let $b = 2a - 1$.

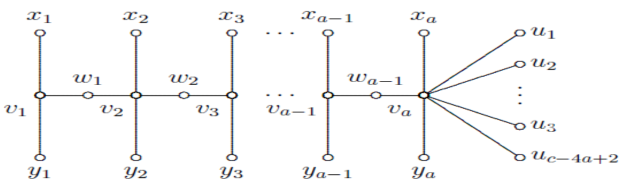

Figure 5: A graph G_5

The set $A = \{ v_i : i = 1,2,\ldots, a \}$ is a γ-set of G, $B = A \cup \{ w_i : i = 1,2,\ldots, a-1 \}$ is a γ_p-set and $C = \{ x_i : i = 1,2,\ldots, a \} \cup \{ y_i : i = 1,2,\ldots, a-1 \} \cup B \cup \{ u_i : i = 1,2,\ldots, c-4a+2 \}$ is a $\tilde{\gamma}_c^p$ - set. Thus, $|V(G)| = a + a + (a + a - 1) + (c - 4a + 2) = c + 1 = n, \gamma(G) = |A| = a, \gamma_p(G) = |B| = a + (a-1) = 2a - 1 = b, and\ \tilde{\gamma}_c^p(G) = |C| = (a) + (a-1) + [a + (a-1)] + (c - 4a + 2) = c$.

This proves the assertion. ■

Corollary 2.3: The difference $\tilde{\gamma}_c^p(G)$-$\gamma_p(G)$ can be made arbitrarily large.

Proof: Let n be a positive integer. By Theorem 2.2, there exists a connected graph G such that $\tilde{\gamma}_c^p(G) = n + 1$ and $\gamma_p(G) = 1$\$. Thus, $\tilde{\gamma}_c^p(G) - \gamma_p(G) = n$, showing that $\tilde{\gamma}_c^p(G) - \tilde{\gamma}_c^p(G)$ can be made arbitrarily large. ■

Theorem 2.4: Let G be a nontrivial connected graph. Then $\tilde{\gamma}_c^p(G) = 1$ if and only if $G = K_1 + H$ where H is a connected graph.

Proof: Suppose that $\tilde{\gamma}_c^p(G) = 1$. Let $S = \{ v \}$ be a $\tilde{\gamma}_c^p$-*set* in G. Since G is nontrivial, $V(G) \setminus S \neq \emptyset$. Since S is outer-connected, $\langle V(G) \setminus S \rangle$ is a connected graph. Let $H = \langle V(G) \setminus S \rangle$. Then $G = K\_1 + H$, where H is a connected graph.

For the converse, suppose that $G = K_1 + H$, where H is a connected graph. Let $S = V(K_1)$. Then S is a dominating set of G. Since each vertex of $V(H) = V(G) \setminus S$ is dominated by exactly one vertex in S, it follows that S is a perfect dominating set. Since $\langle V(G) \setminus S \rangle = H$ is a connected graph, S is an outer-connected dominating set of G. Thus, S is a perfect outer-connected dominating set of G. Hence, $\tilde{\gamma}_c^p(G) = 1$. ■

The next result follows immediately from Theorem 2.4.

Corollary 2.5: Let K_n be a complete graph, $F_n = K_1 + P_n$, and $W_n = K_1 + C_n$.

(i) $\tilde{\gamma}_c^p(K_n) = 1$

(ii) $\tilde{\gamma}_c^p(F_n) = 1$

(iii) $\tilde{\gamma}_c^p(W_n) = 1$

The *join* of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{ uv : u \in V(G), v \in V(H) \}$.

The next result presents a characterization of the perfect outer-connected dominating sets in the join of two connected graphs and gives the corresponding perfect outer-connected domination number.

Theorem 2.6: Let G and H be nontrivial graphs. A proper subset S of $V(G + H)$ is a perfect outer-connected dominating set of $G + H$ if and only if one of the following holds.

(i) $S = \{ x \}$ is a dominating set of G.

(ii) $S = \{ y \}$ is a dominating set of H.

(iii) $S = \{ x \} \cup \{ y \}$, where x is an isolated vertex of G and y is an isolated vertex of H.

Proof: Suppose that $S \subset V(G + H)$ is a perfect outer-connected dominating set in $G + H$. If $S \cap V(H) = \emptyset$, then $S \subset V(G)$.

Suppose that $|S| \geq 2$. Let $x, y \in S$. Then there exists $u \in V(H)$, that is, $u \in V(G + H) \setminus S$ such that $xu, yu \in E(G + H)$. This contradict to our assumption that S is a perfect dominating set of $G + H$. Thus, $|S| \ngeq 2$ and hence $|S| = 1$. Let $S = \{ x \}$. Then S is a dominating set of G. This proves statement (i). Similarly, if $S \subset V(H)$, then statement (ii) holds. Now, suppose that $S_G = S \cap V(G) \neq \emptyset$ and $S_H = S \cap V(H) \neq \emptyset$. Then, $S = S_G \cup S_H$, where $S_G \subset V(G)$ and $S_H \subset V(H)$. This implies that $|S\_G| = 1$ and $|S_H| = 1$ by (i) and (ii) respectively (otherwise S is not a perfect dominating set). Let $S_G = \{ x \}$ and $S_H = \{ y \}$. Suppose that there exists $v \in V(G)$ such that $xv \in E(G)$ and hence $xv \in E(G + H)$. Then $yv \in E(G + H)$. Since $S = \{ x, y \}$, it follows that S is not a perfect dominating set of $G + H$ contrary to our assumption. Thus, x must be an isolated vertex of G. Similarly, y must be an isolated vertex of H (otherwise S is not a perfect dominating set). This proves statement (iii).

For the converse, suppose that statement (i) holds. Then S is a perfect dominating set in $G+H$. Since G and H are a nontrivial graphs, let $v \in V(G) \setminus S$ and $u, y \in V(H)$. Then $v, u, y \in V(G+H) \setminus S$ where $vu, vy \in E(G+H)$. Since v, u and y are arbitrary elements of $V(G+H) \setminus S$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Hence $S = \{x\}$ is a perfect outer connected dominating set of $G+H$. Similarly, if statement (ii) holds, then $S = \{y\}$ is a perfect outer connected dominating set of $G+H$. Now, suppose that statement (iii) holds. Then, $S = \{x, y\}$ is a dominating set $G+H$. Since x is an isolated vertex of G and H is nontrivial, there exists $u \in V(H) \setminus \{y\}$ such that $xu \in E(G+H)$ and since y is an isolated vertex of H and G is nontrivial, there exists $v \in V(G) \setminus \{x\}$ such that $yv \in E(G+H)$ for all $v \in V(G)$. Since $u, v \in V(G+H) \setminus S$, it follows that for each vertex $w \in V(G+H) \setminus S$ there exist exactly one vertex $z \in S$ such that $wz \in E(G+H)$. Hence S is a perfect dominating set of G. Now, since $uv \in E(G+H)$ for all $u, v \in V(G+H) \setminus S$ where $v \in V(G) \setminus \{x\}$ and $u \in V(H) \setminus \{y\}$, it follows that $\langle V(G+H) \setminus S \rangle$ is connected. Hence S is an outer-connected dominating set of $G+H$. Accordingly, S is a perfect outer-connected dominating set of $G+H$. ■

The next result follows immediately from Theorem 2.6.

Corollary 2.7: Let G and H be nontrivial graphs. Then

$$\tilde{\gamma}_c^p(G+H) = \begin{cases} 1, & if\ \gamma(G) = 1\ or\ \gamma(H) = 1 \\ 2, & if\ each\ G\ and\ H\ has\ isolated\ vertex \end{cases}$$

A graph G is connected if there is at least one path that connects every two vertices $x, y \in V(G)$, otherwise, G is disconnected. A dominating set $S \subseteq V(G)$ is called a connected dominating set of G if the subgraph $\langle S \rangle$ induced by S is connected. A set $S \subseteq V(G)$ is a doubly connected dominating set if it is dominating and both $\langle S \rangle$ and $\langle V(G) \setminus S \rangle$ are connected.

Remark 2.8: Every doubly connected dominating set of G is an outer-connected dominating set.

The following result is needed for the characterization of the perfect outer-connected dominating sets of the corona of graphs.

Theorem 2.9 [1]: Let G be connected graph and H be any graph. Then a nonempty set $C \subset V(G \circ H)$ is a $dcd\text{-}set$ of $G \circ H$ if and only if there exists a vertex v of G such that

$$C = V(G) \cup [\cup_{u \in V(G) \setminus \{v\}} V(H^u)] \cup (V(H^v) \setminus T^v)$$

where $\langle T^v \rangle$ is a connected subgraph of H^v.

Lemma 2.10: Let G be connected graph and H be any graph. Then a nonempty proper subset S of $V(G \circ H)$ is a perfect outer-connected dominating set of $G \circ H$ if there exists a vertex v of G such that

$$S = V(G) \cup [\cup_{u \in V(G) \setminus \{v\}} V(H^u)] \cup (V(H^v) \setminus T^v)$$

where $\langle T^v \rangle$ is a connected subgraph of H^v.

Proof: Suppose that there exists a vertex v of G such that

$$S = V(G) \cup [\cup_{u \in V(G) \setminus \{v\}} V(H^u)] \cup (V(H^v) \setminus T^v)$$

where $\langle T^v \rangle$ is a connected subgraph of H^v. Then by Theorem 2.9, S is a doubly connected dominating set of $G \circ H$. By Remark 2.8, S is an outer connected dominating set of G. Further, since $V(G \circ H) \setminus \neq \emptyset$, let $x \in V(G \circ H) \setminus S$. Then $x \in T^v$ for some $v \in V(G)$, that is $xv \in E(G \circ H)$. Since $V(G) \subset S, v \in S$. This implies that each $x \in V(G \circ H) \setminus S$ is dominated by exactly one vertex $v \in S$, and hence, S is a perfect dominating set of $G \circ H$. Accordingly, S is a perfect outer-connected dominating set of $G \circ H$. ■

Lemma 2.11: Let G and H be connected graphs. Then a nonempty proper subset S of $V(G \circ H)$ is a perfect outer-connected dominating set of $G \circ H$ if for every vertex $v \in V(G)$, $S = \bigcup_{v \in V(G)} S_v$ where $S_v = \{x\}$ is a dominating set of H^v.

Proof: Suppose that for every vertex v of G, $S = \bigcup_{v \in V(G)} S_v$ where $S_v = \{x\}$ is a dominating set of H^v.

Then for each $v \in V(G)$, $\{x\}$ is a dominating set of $v + H^v$. Further, each $u \in V(v + H^v) \setminus \{x\}$ is dominated by x, and so $\{x\}$ is a perfect dominating set of $v + H^v$ for each $v \in V(G)$. Now, $S = \bigcup_{v \in V(G)} S_v$ is a perfect dominating set of $\langle \bigcup_{v \in V(G)} V(v + H^v) \rangle = G \circ H$. Clearly, if H is trivial, then $\langle V(G \circ H) \setminus S \rangle$ is connected and hence S is a perfect outer-connected dominating set of $G \circ H$. Suppose that H is nontrivial. Since $V(G \circ H) \setminus S \neq \emptyset$, let $y \in V(G \circ H) \setminus S$. If $y \in V(G)$, then $yu \in E(G \circ H)$ for some $u \in V(H^y) \setminus \{x\}$. If $y \in V(H^v) \setminus \{x\}$ for each $v \in V(G)$, then $yv \in E(G \circ H)$. Since $y, u, v \in V(G \circ H) \setminus S$, it follows that $\langle V(G \circ H) \setminus S \rangle$ is connected and hence S is a perfect outer-connected dominating set of $G \circ H$. ■

Theorem 2.12: Let G be connected graph and H be any graph. Then a nonempty proper subset S of $V(G \circ H)$ is a perfect outer-connected dominating set of $G \circ H$ if and only if one of the following statements is satisfied:

(i) there exists a vertex v of G such that $S = V(G) \cup [\bigcup_{u \in V(G)\setminus\{v\}} V(H^u)] \cup (V(H^v) \setminus T^v)$ where $\langle T^v \rangle$ is a connected subgraph of H^v.

(ii) for every vertex $v \in V(G)$, $S = \bigcup_{v \in V(G)} S_v$ where $S_v = \{x\}$ is a dominating set of H^v.

Proof: Suppose that a nonempty proper subset S of $V(G \circ H)$ is a perfect outer-connected dominating set of $G \circ H$. Consider the following cases:

$Case 1$. Consider that H is not a connected graph.

Let $\langle T \rangle$ be a connected subgraph of H. Then $S = V(G \circ H) \setminus T^v$ for some $v \in V(G)$ is an outer-connected dominating set of $G \circ H$. Thus, there exists $v \in V(G)$ such that

$$S = V(G\backslash circ\ H) \setminus T\hat{}v = V(G) \cup [\bigcup_{u \in V(G)\setminus\{v\}} V(H^u)] \cup (V(H^v) \setminus T^v)$$

where $\langle T^v \rangle$ is a connected subgraph of H^v. This shows statement (i).

$Case 2$. Consider that H is a connected graph.

Then H is a connected subgraph of $G \circ H$ and hence $S = V(G \circ H) \setminus V(H^v)$ for some $v \in V(G)$ is an outer connected dominating set of $G \circ H$. Thus there exists $v \in V(G)$ such that

$$S = V(G \circ H) \setminus T\hat{}v = V(G) \cup [\bigcup_{u \in V(G)\setminus\{v\}} V(H^u)] \cup (V(H^v) \setminus T^v)$$

where $\langle T^v \rangle$ is a connected subgraph of H^v. Again, this shows statement (i). Now, if $S_v = \{x\}$ is a dominating set of H^v for each $v \in V(G)$, then $S = \bigcup_{v\in V(G)} S_v$ showing statement (ii).

For the converse, suppose that statements (i) or (ii) holds. Consider first statement (i). In view of Lemma 2.10, S is a perfect outer connected dominating set of $G \circ H$. Next, consider statement (ii), then S is a perfect outer connected dominating set of $G \circ H$ by Lemma 2.11. ■

The following corollaries are immediate consequences of Theorem 2.12.

Corollary 2.13: Let G and H be connected graphs. Then there exists $v \in V(G)$ such that $\tilde{\gamma}_c^p(G \circ H) = |V(G)| + \sum_{u\in V(G)\setminus\{v\}} |V(H^u)|$.

Corollary 2.14: Let G be connected graph of order $n \geq 1$ and H be any graph of order m. Then $\tilde{\gamma}_c^p(G \circ H) = mn + n - r$ where $r = max\{|V(J)|: J\ is\ a\ connected\ subgraph\ of\ H\}$. In particular, if H is connected, then $\tilde{\gamma}_c^p(G \circ H) = mn + n - m$.

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