# Inverse Clique Domination in Graphs

Carmelita Loquias[1], Enrico Enriquez[1] and Jonecis Dayap[2]

### Abstract

*Let G be a connected simple graph. A nonempty subset S of the vertex set V (G) is a clique in G if the graph hSi induced by S is complete. A clique S in G is a clique dominating set if it is a dominating set. Let C be a minimum clique dominating set in G. The clique dominating setS ⊆ V (G)\ C is called an inverse clique dominating set with respect toC. The minimum cardinality of inverse clique dominating set is calledan inverse clique domination number of G and is denoted by γcl−1(G). An inverse clique dominating set of cardinality γcl−1(G) is called γcl−1-set of G. In this paper we investigate the concept and give some important results.*

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University of San Carlos[1]
University of San Jose Recoletos[2]

# 1 Introduction

Let $G$ be a simple connected graph. A subset $S$ of a vertex set $V(G)$ is a *dominating set* of $G$ if for every vertex $v \in (V(G)\backslash S)$, there exists a vertex $\cancel{x \in S \text{ such that } xv \text{ is an edge}}$ of $G$. The *domination number* $\gamma(G)$ of $G$ is the smallest cardinality of a dominating set $S$ of $G$. Dominating sets have several applications in a variety of fields, including communication and electrical networks, protection and location strategies, data structures and others. For more background on dominating sets, the reader may refer to [1, 2].

A complete graph of order $n$, denoted by $K_n$, is the graph in which every pair of its distinct vertices are joined by an edge. A nonempty subset $S$ of $V(G)$ is a clique in $G$ if the graph $\langle S\rangle$ induced by $S$ is complete. A nonempty subset $S$ of a vertex set $V(G)$ is a *clique dominating set* of $G$ if $S$ is a dominating set and $S$ is a clique in $G$. The minimum cardinality among all clique dominating sets of $G$, denoted by $\gamma_{cl}(G)$, is called the clique domination number of $G$. A clique dominating set $S$ of $G$ with $|S| = \gamma_{cl}(G)$ is called a $\gamma_{cl}$-*set* of $G$. Clique dominating sets have a great diversity of applications. In setting up the communications links in a network one might want a strong core group that can communicate with each other member of the core group and so that everyone outside the core group could communicate with someone within the core group. A group of forest fire sentries that could see various sections of a forest might also be positioned in such a way that each could see the others in order to use triangulation to locate the site of a fire. In addition, the properties of dominating sets are useful in identifying structural properties of a social network [4].

Wolk [3] presents a forbidden subgraph characterization of a class of graphs which have a dominating clique of size one. He called such a dominating clique a central vertex or central point. The idea of Wolk was extended by Cozzens and Kelleher [5] to get forbidden subgraph conditions sufficient to imply the existence of a dominating set that induces a complete subgraph, a dominating clique. Daniel and Canoy [7] characterized the clique dominating sets in the join, corona, composition and Cartesian product of graphs and determine the corresponding clique domination number of the resulting graph.

The concept of dominating sets introduced by Ore and Berge [6], is currently receiving much attention in the literature of graph theory. Several types of domination parameters have been studied by imposing several conditions on dominating sets. Ore observed that the complement of every minimal dominating set of a graph with minimum degree at least one is also a dominating set. This implies that every graph with minimum degree at least one has two disjoint dominating sets. Recently several authors initiated the study of the cardinality of pairs of disjoint dominating sets in graphs. The inverse domination number is the minimum cardinality of a dominating set whose complement contains a minimum dominating set.

Motivated by the inverse domination number and the notion of clique domination number, we introduce a variant of domination in graphs - the inverse clique domination in graphs. Let $C$ be a minimum clique dominating set in $G$. The clique dominating set $S \subseteq V(G) \setminus C$ is called an *inverse clique dominating set* with respect to $C$. The minimum cardinality of inverse clique dominating set is called an *inverse clique domination number* of $G$ and is denoted by $\gamma_{cl}^{-1}(G)$. An inverse clique dominating set of cardinality $\gamma_{cl}^{-1}(G)$ is called $\gamma_{cl}^{-1}$-*set* of $G$. In this paper we investigate the concept and give some important results. Throughout this paper the notations $K_n$, $P_n$, and $C_n$, denotes the complete graph, path graph, and cycle graph respectively of $n$ distinct vertices. Unless otherwise stated, all subsets of the vertex set of a graph are assumed to be nonempty.

# 2 Results

Since $\gamma_{cl}^{-1}(G)$ does not always exists in a connected nontrivial graph $G$, we denote by $\mathcal{CL}^{-1}(G)$ a family of all graphs with inverse clique dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{CL}^{-1}(G)$. From the definitions, the following remarks are immediate.

**Remark 2.1** *Let $G$ be a nontrivial connected graph. If $C$ is a $\gamma_{cl}$-set and $S$ is an inverse clique dominating set of $G$ with respect to $C$, then $C \cap S = \varnothing$.*

**Remark 2.2** *Let $G$ be a connected graph of order $n \geq 2$. Then*

*(i)* $1 \leq \gamma_{cl}^{-1}(G) \leq n/2$, *and*

*(ii)* $\gamma(G) \leq \gamma_{cl}(G) \leq \gamma_{cl}^{-1}(G)$.

The next result says that the value of the parameter $\gamma_r^{-1}(G)$ ranges over all positive integers.

**Theorem 2.3 (Realization Problem)** *Given positive integers $k$ and $n$ such that $n \geq 2$ and $1 \leq k \leq n/2$ , there exists a connected nontrivial graph $G$ with $|V(G)| = n$ and $\gamma_{cl}^{-1}(G) = k$.*

**Proof.** Consider the following cases:

*Case*1. Suppose $k = 1$.
Let $G = K_n$. Then, clearly, $|V(G)| = n$ and $\gamma_s^{-1}(G) = 1$.

*Case*2. Suppose $k = 2$.
Let $H_1 = P_4 = [x_1, x_2, x_3, x_4]$ and consider the graph $G = K_1 + H_1$. Then the set $C = V(K_1)$ is a $\gamma_{cl}$-*set* of $G$ and the set $S = \{x_2, x_3\}$ is a $\gamma_{cl}^{-1}$-*set* of $G$. Thus, $|V(G)| = 5 = n$ and $\gamma_{cl}^{-1}(G) = 2 = k$.

*Case*3. Suppose $3 \leq k < n/2$.

Let $H_2 = K_1 + P_m$ where $P_m = [x_1, x_2, ..., x_m]$ $(m \geq 5)$ and $K_1 = [y]$ and let $K_{k-2} = [v_1, v_2, ..., v_{k-2}]$. Consider the graph $G$ obtained from $H_2$ by adding the edges $yv_1, yv_2, ...yv_{k-2}$, and $yx_1, yx_2, ..., yx_m$, and $x_3v_1, x_3v_2, ...x_3v_{k-2}$, and $x_4v_1, x_4v_2, ...x_4v_{k-2}$, and $v_2x_6, v_3x_8, ..., v_{k-2}x_{m-1}$ (see Figure 1).

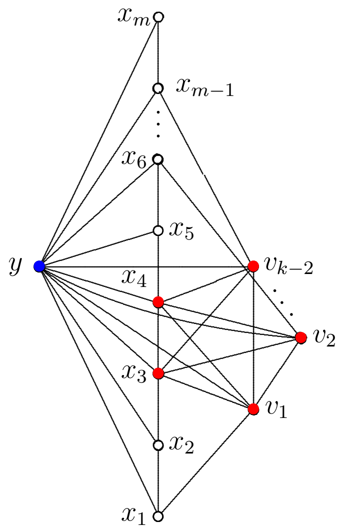


Figure 1: A graph $G$ with $\gamma_{cl}^{-1}(G) = k$

Let $m = 2k - 1$. Then the set $C = \{y\}$ is a $\gamma_{cl}$-*set* of $G$ and the set $S = \{x_3, x_4, v_1, v_2, ..., v_{k-2}\}$ is a $\gamma_{cl}^{-1}$-*set* of $G$. Thus, $\gamma_{cl}^{-1}(G) = |S| = 2 + (k-2) = k$. Now,

$$|V(G)| = 1 + m + (k-2) = 1 + (2k-1) + (k-2) = 3k - 2 = n.$$

Since

$$\begin{aligned} 3k - 2 = n \quad &\Leftrightarrow \quad \frac{3k-2}{2} = \frac{n}{2} \\ &\Leftrightarrow \quad \frac{3k}{2} = \frac{n}{2} + 1 \\ &\Leftrightarrow \quad k + \frac{k}{2} = \frac{n}{2} + 1, \end{aligned}$$

it follows that $k < n/2$ considering that $\dfrac{k}{2} > 1$ whenever $k \geq 3$.

*Case*4. Suppose $k = n/2$.

Let $V(K_k) = \{x_1, x_2, ..., x_k\}$ and $V(K_m) = \{y_1, y_2, ..., y_m\}$ $(m = k)$. Consider the graph $G$ obtained from $K_k$ and $K_m$ by adding the edges $x_1y_1, x_2y_2, ...x_ky_m$ (see Figure 2).

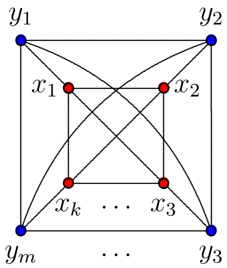


Figure 2: A graph $G$ with $\gamma_{cl}^{-1}(G) = n/2$

The set $C = \{x_1, x_2, ..., x_k\}$ is a $\gamma_{cl}$-*set* and $S = \{y_1, y_2, ..., y_m\}$ is a $\gamma_{cl}^{-1}$-*set* of $G$. Thus, $|V(G)| = k + m = k + k = 2k = n$ and $\gamma_r^{-1}(G) = k = n/2$.

This proves the assertion. $\square$

**Corollary 2.4** *The difference $\gamma_{cl}^{-1}(G) - \gamma_{cl}(G)$ can be made arbitrarily large.*

*Proof*: Let $k$ be a positive integer. By Theorem 2.3, there exists a connected graph $G$ such that $\gamma_{cl}^{-1}(G) = k+1$ and $\gamma_{cl}(G) = 1$. Thus, $\gamma_{cl}^{-1}(G) - \gamma_{cl}(G) = k$. Therefore, $\gamma_{cl}^{-1} - \gamma_r$ can be made arbitrarily large. $\square$

**Theorem 2.5** *Let $G$ be a nontrivial connected graph. Then $\gamma_{cl}^{-1}(G) = 1$ if and only if $G = K_1 + H$ where $\gamma(H) = 1$.*

*Proof*: Suppose that $\gamma_{cl}^{-1}(G) = 1$. Let $S = V(K_1)$ be a $\gamma_{cl}^{-1}$-*set* of $G$. Set $V(H) = V(G) \setminus S$. Since $\gamma_{cl}(G) \leq \gamma_{cl}^{-1}(G) = 1$ by Remark 2.2, it follows that $\gamma_{cl}(G) = 1$. Let $C = \{x\}$ be a $\gamma_{cl}$-*set* of $G$. Since $C \cap S = \varnothing$ by Remark 2.1, $C \subset V(H)$, that is, $\gamma(H) = 1$. Therefore, $G = K_1 + H$ where $\gamma(H) = 1$.

For the converse, suppose that $G = K_1 + H$ where $\gamma(H) = 1$. Let $C = V(K_1) = \{x\}$ be a $\gamma_{cl}$-*set* of $G$ and let $S = \{y\}$ be a dominating set of $H$. Since $S$ is a clique dominating set of $G$ and $C \cap S = \varnothing$, $S \subseteq (V(G) \setminus C)$, that is, $S$ is a $\gamma_{cl}^{-1}$-*set* of $G$. Hence, $\gamma_{cl}^{-1}(G) = 1$. $\square$

The following results are consequences of Theorem 2.5

**Corollary 2.6** *Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_{cl}^{-1}(G) = 1$ if and only if $G = K_2 + H$ for some subgraph $H$.*

**Corollary 2.7** *If $G$ is a complete graph of order $n \geq 2$, then $\gamma_{cl}^{-1}(G) = 1$.*

The next result is a characterization of an inverse clique dominating set of a nontrivial connected graph $G = K_1 + H$.

**Theorem 2.8** *Let $G = K_1 + H$. Then a subset $S$ of $V(G)$ is an inverse clique dominating set of $G$ if and only if $S$ is a clique dominating set of $H$.*

*Proof*: Suppose that $S$ is an inverse clique dominating set of $G$. Then $S$ clique dominating set of $G$. Since $G = K_1 + H$, $V(K_1)$ is a $\gamma_{cl}$-*set* of $G$. This implies that $S \subset V(G) \setminus V(K_1) = V(H)$. Thus $S$ is a clique dominating set of $H$.

For the converse, suppose that $S$ is a clique dominating set of $G$. Since $G = K_1 + H$, $V(K_1)$ is a $\gamma_{cl}$-*set* of $G$. This implies that $S \subset V(G) \setminus V(K_1)$ is an inverse clique dominating set of $G$ with respect to $C = V(K_1)$. □

The following is a quick consequence of Theorem 2.8.

**Corollary 2.9** *Let $G = K_1 + H$. Then $\gamma_{cl}^{-1}(G) = \gamma_{cl}(H)$.*

*Proof*: Suppose that $G = K_1 + H$. If $S$ is an inverse clique dominating set of $G$ then $S$ is a clique dominating set of $H$ by Theorem 2.8. This implies that,

$$\begin{aligned} \gamma_{cl}^{-1}(G) &\leq |S| \text{ for all clique dominating set } S \text{ of } H \\ &\leq min\{|S| : S \text{ is a clique dominating set of } H\} = \gamma_{cl}(H). \end{aligned}$$

Further, if $S$ be a clique dominating set of $H$ then $S$ is an inverse clique dominating set of $G$ by Theorem 2.8. This implies that

$$\begin{aligned} \gamma_{cl}(H) &\leq |S| \text{ for all inverse clique dominating set } S \text{ of } G \\ &\leq min\{|S| : S \text{ is an inverse clique dominating set of } G\} = \gamma_{cl}^{-1}(G). \end{aligned}$$

Thus, $\gamma_{cl}^{-1}(G) = \gamma_{cl}(H)$. □

The join of two graphs $G$ and $H$, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set

$$E(G + H) = E(G) \cup E(H) \cup uv : u \in V(G), v \in V(H).$$

We are needing the following result for the characterization of the inverse clique dominating set of the join of two graphs.

**Theorem 2.10** [7] *Let $G$ and $H$ be any two graphs. A subset $S$ of $V(G + H)$ is a clique dominating set of $G+H$ if and only if one of the following statements holds:*

*(i) $S$ is a clique dominating set of $G$.*

*(ii) $S$ is a clique dominating set of $H$.*

*(iii) $S = S1 \cup S2$, where $\langle S1\rangle$ and $\langle S2\rangle$ are cliques in $G$ and $H$, respectively.*

A clique dominating $S$ is a *secure clique dominating* set in $G$ if for every $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a clique dominating set in $G$. The secure clique domination number of $G$, denoted by $\gamma_{scl}(G)$, is the smallest cardinality of a secure clique dominating set in $G$.

The following result is the characterization of the inverse clique dominating sets in the join of two graphs.

**Theorem 2.11** *Let $G$ and $H$ be any two graphs. A subset $S$ of $V(G+H)$ is an inverse clique dominating set of $G+H$ if and only if $|S| \geq |C|$ for some clique dominating set $C$ of $G+H$ and one of the following statements holds:*

*$(i)$ $S$ is a clique dominating set of $G$.*

*$(ii)$ $S$ is a clique dominating set of $H$.*

*$(iii)$ $S = S1 \cup S2$, where $\langle S1 \rangle$ and $\langle S2 \rangle$ are cliques in $G$ and $H$, respectively.*

*Proof*: Suppose that a subset $S$ of $V(G+H)$ is an inverse clique dominating set of $G+H$. Then let $C$ be a $\gamma_{cl}$-*set* of $G+H$. This implies that $|C| \leq |S|$. Further, $S$ is a clique dominating set of $G+H$. Thus, statement $(i)$ or $(ii)$ or $(iii)$ holds by Theorem 2.10.

For the converse, suppose that statement $(i)$ or $(ii)$ or $(iii)$ holds. Then $S$ is a clique dominating set of $G+H$ by Theorem 2.10. Suppose first that $(i)$ holds. Since $|S| \geq |C|$ for some clique dominating set $C$ of $G+H$, let $C$ be a $\gamma_{cl}$-*set* such that $S \cap C = \varnothing$. Then $S \subseteq V(G+H) \setminus C$ implies that $S$ is an inverse clique dominating set of $G+H$. Similary, if $(ii)$ holds then $S$ is an inverse clique dominating set of $G+H$. Now, suppose that $(iii)$ holds. Since $\langle S1 \rangle$ and $\langle S2 \rangle$ are cliques in $G$ and $H$ respectively, it follows that $S = S1 \cup S2$ is a clique dominating set of $G+H$. Following similar arguments used in $(i)$ we conclude that $S$ is an inverse clique dominating set of $G+H$. □

**Corollary 2.12** *Let $G$ and $H$ be nontrivial graphs. Then*

$$\gamma_{cl}^{-1}(G+H) = \begin{cases} 1, \text{ if } (\gamma(G)=1 \text{ and } \gamma(H)=1) \text{ or} \\ \qquad (\gamma_{scl}(G) \geq 2 \text{ or } \gamma_{scl}(H) \geq 2) \\ 2, \text{ if } (\gamma(G) \neq 1 \text{ or } \gamma(H) \neq 1) \end{cases}$$

*Proof*: Suppose that $\gamma(G) = 1$ and $\gamma(H) = 1$. Let $S = \{a\}$ and $C = \{b\}$ be dominating sets in $G$ and $H$ respectively. Then $G = \langle S \rangle + J$ and $H = \langle C \rangle + I$ for some subgraphs $J$ and $I$ of $G$ and $H$ respectively. Thus, $G+H = (\langle S \rangle + J) + (\langle C \rangle + I) = \langle S \rangle + (\langle C \rangle + J + I) = K_1 + (K_1 + J + I)$ where $\gamma(K_1 + J + I) = 1$. Thus $\gamma_{cl}^{-1}(G+H) = 1$ by Theorem 2.5.

Suppose that $\gamma_{scl}(G) \geq 2$. Let $A$ be a $\gamma_{scl}$-*set* in $G$ and let $x, y \in A$ $(x \neq y)$. Then $S = \{x\}$ and $C = \{y\}$ are dominating sets of $G$ (otherwise $A$ is not a secure clique dominating set of $G$) and hence of $G + H$. Let $C$ be a minimum clique dominating set of $G + H$. Then $S \subseteq V(G + H) \setminus C$ is an inverse clique dominating set of $G + H$ with respect to $C$. Hence $\gamma_{cl}^{-1}(G + H) = 1$. Similarly, if $\gamma_{scl}(H) \geq 2$ , then $\gamma_{cl}^{-1}(G + H) = 1$.

Suppose that $\gamma(G) \neq 1$ and $\gamma(H) = 1$. Let $C = \{b\}$ be the only dominating set of $H$ and hence of $G + H$. Then $C$ is a minumum clique dominating set of $G + H$. Since $S = \{x, y\}$ is a clique dominating set of $G + H$ for any $x \in V(G)$ and for any $y \in V(H) \setminus \{b\}$, it follows that $S \subseteq V(G+H) \setminus C$ is an inverse clique dominating set of $G+H$ with respect to $C$. Thus, $2 = |S| \geq \gamma_{cl}^{-1}(G+H)$. Since $\gamma(G) \neq 1$, it follows that $\gamma(\langle V(G + H) \setminus C\rangle) \neq 1$ and hence $\gamma_{cl}^{-1}(G + H) \neq 1$. Thus, $\gamma_{cl}^{-1}(G + H) \geq 2$. Therefore $\gamma_{cl}^{-1}(G + H) = 2$. Similarly, if $\gamma(G) = 1$ and $\gamma(H) \neq 1$ then $\gamma_{cl}^{-1}(G + H) = 2$. Suppose that $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. Then $\gamma(G + H) \neq 1$. Since $G$ and $H$ are nontrivial graphs, let $x, v \in V(G)$ and $y, u \in V(H)$, that is, $xy, vu \in E(G + H)$. Then $C = \{x, y\}$ and $S = \{v, u\}$ are minimum clique dominating sets of $G + H$. Since $C \cap S = \varnothing$, it follows that $S \subseteq V(G + H) \setminus C$ is a minimum inverse clique dominating set of $G + H$ with respect to C. Thus, $\gamma_{cl}^{-1}(G + H) = 2$. □

The *composition* of two graphs $G$ and $H$ is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x, u)(y, v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

A subset $C$ of $V(G[H]) = V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$. We shall use this form to denote any subset $C$ of $V(G[H]) = V(G) \times V(H)$.

**Corollary 2.13** *If $G$ and $H$ are complete nontrivial graphs, then*

$$\gamma_{cl}^{-1}(G[H]) = 1.$$

*Proof*: Since $G[H]$ is a complete graph, $G[H] = K_1 + J$ where $\gamma(J) = 1$. By Theorem 2.5 $\gamma_{cl}^{-1}(G[H]) = 1$. □

**Theorem 2.14** [7] *Let $G$ and $H$ be connected nontrivial graphs. Then $G[H]$ has a clique dominating set if and only if $G$ has a clique dominating set.*

**Lemma 2.15** *Let $G$ and $H$ be connected nontrivial graphs. Then $G[H]$ has an inverse clique dominating set if and only if $G$ has a clique dominating set of order $k \geq 2$. Furthermore, if $G$ has a clique dominating set of order $k = 1$, then $G$ has disjoint clique dominating sets (and $H$ has a clique dominating set) or $H$ has disjoint clique dominating sets.*

*Proof*: Suppose that $G[H]$ has an inverse clique dominating set. Then $G[H]$ has a clique dominating set by definition. Thus, $G$ has a clique dominating set of order $k \geq 2$ by Theorem 2.14. Furthermore, suppose that $S = \{x\}$ is the only clique dominating set in $G$ and $H$ has only one clique dominating set. If $T_x$ is not a dominating set in $H$, then $C = \{x\} \times T_x$ is not a dominating set in $G[H]$ contrary to our assumption. If $T_x$ is a dominating set in $H$, then $C$ is a clique dominating set in $G[H]$. Since $T'_x$ is not a clique dominating set in $H$ for all $T'_x \subseteq V(H) \setminus T_x$, it follows that $C' = \{x\} \times T'_x$ is not a clique dominating set in $G[H]$. This implies that $G[H]$ has no inverse clique dominating set contrary to our assumption. Thus, $S = \{x\}$ is a clique dominating set in $G$ and $G$ has more than one clique dominating sets or $H$ has more than one clique dominating sets. First, let $S'$ be a clique dominating set in $G$ and let $T_x$ be a minimum clique dominating set in $H$. Then $C = S \times T_x = \{x\} \times T_x$ is a minimum clique dominating set in $G[H]$ and $C' = S' \times T_x$ clique dominating set in $G[H]$. Since $G[H]$ has an inverse clique dominating set, let $C'$ be the inverse clique dominating set in $G[H]$ with respect to $C$. This implies that $\varnothing = C \cap C' = (S \times T_x) \cap (S' \times T_x) = (S \cap S') \times T_x$. Since $T_x \neq \varnothing$, it follows that $S \cap S' = \varnothing$. Thus, $G$ has disjoint clique dominating sets. Next, let $T_x$ be a minimum clique dominating set in $H$ and $T'_x$ be clique dominating sets in $H$. Then $C = S \times T_x = \{x\} \times T_x$ is a minimum clique dominating set in $G[H]$ and $C' = \{x\} \times T'_x$ is an inverse clique dominating sets in $G[H]$ with respect to $C$ by assumption. This implies that $\varnothing = C \cap C' = (S \times T_x) \cap (S \times T'_x) = S \times (T_x \cap T'_x)$. Since $S \neq \varnothing$, it follows that $T_x \cap T'_x = \varnothing$. Thus, $H$ has disjoint clique dominating sets.

For the converse, suppose that $G$ has a clique dominating set of order $k \geq 2$. Let $S$ be a clique dominating set in $G$. Further, $G[H]$ has a clique dominating set by Theorem 2.14. Let $a \in V(H)$ such that $C = S \times \{a\}$ is a clique dominating set in $G[H]$. Since $H$ is nontrivial, $V(H) \setminus \{a\} \neq \varnothing$. Let $b \in V(H) \setminus \{a\}$. Clearly, $C' = S \times \{b\}$ is a clique set in $G[H]$. Now,

$$\begin{aligned} N_{G[H]}[C'] &= N_{G[H]}[S \times \{b\}] \\ &= N_{G[H]}(S \times \{b\}) \cup (S \times \{b\}). \end{aligned}$$

Since $G$ and $H$ are connected graphs,

$$N_{G[H]}(S \times \{b\}) = [N_G(S) \times V(H)] \cup [(S \times V(H)) \setminus (S \times \{b\})].$$

Thus,

$$\begin{aligned} N_{G[H]}[C'] &= [N_G(S) \times V(H) \cup (S \times V(H)) \setminus (S \times \{b\})] \cup (S \times \{b\}) \\ &= [N_G(S) \times V(H)] \cup [(S \times V(H)) \setminus (S \times \{b\})] \cup (S \times \{b\}) \\ &= [N_G(S) \times V(H)] \cup [S \times V(H)] \\ &= [N_G(S) \cup S] \times V(H)] \\ &= N_G[S] \times V(H). \end{aligned}$$

Since $S$ is a dominating set in $G$, $N_G[S] = V(G)$ and so,

$$N_{G[H]}[C'] = V(G) \times V(H) = V(G[H]).$$

Therefore, $C'$ is also a clique dominating set in $G[H]$. Finally, if $C$ is a minimum clique dominating set in $G[H]$, then $C' \subseteq V(G[H]) \setminus C$ is an inverse clique dominating set in $G[H]$ with respect to $C$. If $C$ is not a minimum clique dominating set in $G[H]$, then there exists $D \subset V(G[H])$ such that $D$ is a minimum clique dominating set in $G[H]$ and $D \cap C' = \varnothing$. Hence $C' \subseteq V(G[H]) \setminus D$ is an inverse clique dominating set in $G[H]$ with respect to $D$.

Furthermore, suppose that $G$ has a clique dominating set of order $k = 1$. Let $S = \{x\}$ be a minimum clique dominating set in $G$. First, if $G$ has disjoint clique dominating sets (and $H$ has a clique dominating set), then let $S'$ be a clique dominating set in $G$ with $S \cap S' = \varnothing$. Let $T_x$ be a clique dominating set in $H$. Then $C = S \times T_x = \{x\} \times T_x$ is a minimum clique dominating set in $G[H]$ and $C' = S' \times T_x$ is a clique dominating set in $G[H]$. Now,

$$C \cap C' = (S \times T_x) \cap (S' \times T_x) = (S \cap S') \times T_x = \varnothing \cap T_x = \varnothing.$$

This implies that $C' \subseteq (V(G[H]) \setminus C)$ is an inverse clique dominating set in $G[H]$ with respect to $C$. Next, if $H$ has disjoint clique dominating sets, then let $T_x$ be a minimum clique dominating set in $H$ and $T'_x$ be a clique dominating set in $H$ with $T_x \cap T'_x = \varnothing$. Then $C = S \times T_x = \{x\} \times T_x$ is a minimum clique dominating set in $G[H]$ and $C' = S \times T'_x$ is a clique dominating set in $G[H]$. Now,

$$C \cap C' = (S \times T_x) \cap (S \times T'_x) = S \times (T_x \cap T'_x) = S \cap \varnothing = \varnothing.$$

This implies that $C' \subseteq (V(G[H]) \setminus C)$ is an inverse clique dominating set in $G[H]$ with respect to $C$. $\square$

The following result is the characterization of an inverse clique dominating set of the composition of two graphs.

**Theorem 2.16** *Let $G$ and $H$ be connected nontrivial graphs. A subset $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is an inverse clique dominating set in $G[H]$ if and only if*

*(i) $S$ is a clique dominating set in $G$ such that $T_x$ is a clique set in $H$ for each $x \in S$ with $|S| \geq 2$, or*

*(ii) $S = \{x\}$ is a clique dominating set in $G$ such that $T_x$ and $T'_x$ are clique dominating sets in $H$ with $T_x \cap T'_x = \varnothing$, or*

(*iii*) *$S = \{x\}$ and $S'$ are clique dominating sets in $G$ such that $T_x$ is a clique dominating set in $H$ with $S \cap S' = \varnothing$.*

*Proof*: Suppose that $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is an inverse clique dominating set in $G[H]$. Then $G$ has a clique dominating set of order $k \geq 2$ by Lemma 2.15. Let $S$ be a clique dominating set in G with $|S| \geq 2$. Suppose that $T_x$ is not a clique set in $H$ for some $x \in S$. Then $|T_x| \geq 2$ and there exists $a, b \in T_x$ such that $ab \notin E(H)$. This implies that $(x, a)(x, b) \notin E(G[H])$ for some $x \in S$. Since $(x, a), (x, b) \in C$, it follows that $C$ is not a clique dominating set in $G[H]$ contrary to our assumption. Thus, $T_x$ is a clique set in $H$ for each $x \in S$. This proves statement $(i)$. Now, suppose that $S = \{x\}$. Then $G$ has disjoint clique dominating sets (and $H$ has a clique dominating set) or $H$ has disjoint clique dominating sets by Lemma 2.15. If $H$ has disjoint clique dominating sets, then let $T_x$ and $T'_x$ be clique dominating sets in $H$ with $T_x \cap T'_x = \varnothing$. This proves statement $(ii)$. If $G$ has disjoint clique dominating sets and $H$ has a clique dominating set, then let $S'$ be a clique dominating sets in $G$ such that $T_x$ is a clique dominating set in $H$ with $S \cap S' = \varnothing$. This proves statement $(iii)$.

For the converse, suppose that statement $(i)$, $(ii)$, or $(iii)$ holds. Then $G[H]$ has an inverse clique dominating set by Lemma 2.15. Let $C = \bigcup_{x \in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, be an inverse clique dominating set in $G[H]$. □

**Corollary 2.17** *Let $G$ and $H$ be connected nontrivial graphs. Then*

$$\gamma_{cl}^{-1}(G[H]) = \begin{cases} 1, & \text{if } \exists\, x, y \in V(G) \text{ such that } \{x\} \text{ and } \{y\} \\ & \quad \text{are dominating sets in } G \text{ and } \gamma(H) = 1 \\ & \quad \text{or } \exists\, a, b \in V(H) \text{ such that } \{a\} \text{ and } \{b\} \\ & \quad \text{are dominating sets in } H \text{ and } \gamma(G) = 1 \\ \gamma_{cl}(G), & \text{if } S \text{ is a clique dominating set in } G \text{ with} \\ & \quad |S| \geq 2 \text{ and } T_x \text{ is a clique set in } H \text{ for each } x \in S. \end{cases}$$

*Proof*: Suppose that there exists distinct vertices $x, y \in V(G)$ such that $S = \{x\}$ and $S' = \{y\}$ are dominating sets in $G$ and $\gamma(H) = 1$. Let $T_x = \{a\}$ be a dominating set in $H$. Then $S$ and $S'$ are clique dominating sets in $G$ such that $T_x$ is a clique dominating set in $H$ with $S \cap S' = \varnothing$. This implies that $C = \bigcup_{x \in S} [\{x\} \times T_x] = S \times T_x = \{x\} \times \{a\} = \{(x, a)\}$ is an inverse clique dominating set in $G[H]$ by Theorem 2.16$(iii)$. Thus, $\gamma_{cl}^{-1}(G[H]) \leq |C| = 1$ and so, $\gamma_{cl}^{-1}(G[H]) = 1$. Similarly, if there exists distinct vertices $a, b \in V(H)$ such that $T_x = \{a\}$ and $T'_x = \{y\}$ are dominating sets in $H$ and $\gamma(G) = 1$, then by using Theorem 2.16$(ii)$, $\gamma_{cl}^{-1}(G[H]) = 1$.

Suppose that $S$ is a clique dominating set in $G$ such that $T_x = \{a\}$ is a clique set in $H$ for each $x \in S$ with $|S| \geq 2$. Then $C = \bigcup_{x \in S} [\{x\} \times T_x] = S \times \{a\}$, is an inverse clique dominating set in $G[H]$ by Theorem 2.16$(i)$. Thus,

$$\begin{aligned} \gamma_{cl}^{-1}(G[H]) &= min\{|C| : C \text{ is an inverse clique dominating set in } G[H]\} \\ &= min\{|C| : C = S \times \{a\}\} \\ &= min\{|S| : S \text{ is a clique dominating set in } G\} \\ &= \gamma_{cl}(G). \square \end{aligned}$$

**Corollary 2.18** *Let $G$ and $H$ be connected nontrivial graphs. If $S = \{x\}$ is a clique dominating set in $G$ such that $T_x$ and $T'_x$ are clique dominating sets in $H$ with $T_x \cap T'_x = \varnothing$ then $\gamma_{cl}^{-1}(G[H]) = \gamma_{cl}^{-1}(H)$.*

*Proof*: Suppose that $S = \{x\}$ is a clique dominating set in $G$ such that $T_x$ and $T'_x$ are clique dominating sets in $H$ with $T_x \cap T'_x = \varnothing$. Then $C = \bigcup_{x \in S} [\{x\} \times T_x] = \{x\} \times T_x$, is an inverse clique dominating set in $G[H]$ by Theorem 2.16$(ii)$. Thus,

$$\begin{aligned} \gamma_{cl}^{-1}(G[H]) &= min\{|C| : C \text{ is an inverse clique dominating set in } G[H]\} \\ &= min\{|C| : C = \{x\} \times T_x\} \\ &= min\{|T_x| : T_x \text{ is a clique dominating set in } H, \forall |T_x| \geq \gamma_{cl}(H)\} \\ &= \gamma_{cl}^{-1}(H). \square \end{aligned}$$

## Reference

Cockayne, E. J. & Hedetniemi, S. T. (1977, September). Towards a theory of domination in graphs. *Networks, 7*(3), 247-261.

Cockayne, E. J., Dawes, R. M., & Hedetniemi, S. T. (1980, September). Total domination in graphs. *Networks, 10*(3), 211-219.

Cozzens, M. B. & Kelleher, L. L. (1990, December). Dominating cliques in graphs. *Discrete Mathematics, 86*(1-3), 101 – 116. Retrieved from https://doi.org/10.1016/0012-365X(90)90353-J

Daniel, T. V. & Canoy, S. Jr. (2015). Clique domination in a graph. *Applied Mathematical Sciences, 9*(116), 5749 - 5755.

Kelleher, L. L. & Cozzens, M. B. (1988, December). Dominating sets in social network graphs. *Mathematical Social Sciences, 16*(3), 267-279.

Ore, O. (1962). *Theory of graph.* Providence, RI : American Mathematical Society.

Wolk, E. S. (1965). A note on the comparability graph of a tree. *Proc. Amer. Math. Sot., 16*, 17-20.