Disjoint Secure Dominationin the Join of Graphs

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Abstract

Let $G = (V(G), E(G))$ be a simple connected graph. A dominating set S in G is called a secure dominating set in G if for every $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. The minimum cardinality of secure dominating set is called the securedomination number of G and is denoted by $\gamma_s(G)$. A secure dominatingset of cardinality $\gamma_s(G)$ is called γ_s-set of G. Let D be a minimum secure dominating set in G. The secure dominating set $S \subseteq V(G) \setminus D$ is called an inverse secure dominating set with respect to D. The inversesecure domination number of G denoted by $\gamma_s{-1}(G)$is the minimumcardinality of an inverse secure dominating set in G. An inverse secure dominating set of cardinality $\gamma_s{-1}(G)$ is called $\gamma_s{-1}$-set. A disjoint secure dominating set in G is the set $C = D \cup S \subseteq V(G)$. The disjoint securedomination number of G denoted by $\gamma\gamma_r(G)$is the minimum cardinalityof a disjoint secure dominating set in G. A disjoint secure dominating set of cardinality $\gamma\gamma_s(G)$ is called $\gamma\gamma_s$-set. In this paper, we show that every integers k and n with $k \in \{2, 4, 5, ...n - 1, n\}$ is realizable as disjoint secure domination number, and order of G respectively. Further, we give the characterization of the disjoint secure dominating set in the join of two graphs.

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1 Introduction

The concept of domination in graphs introduced by Claude Berge in 1958 and Oystein Ore in 1962 [10] is currently receiving much attention in literature. Following the article of Ernie Cockayne and Stephen Hedetniemi [3], the domination in graphs became an area of study by many researchers. One type of domination parameter is the secure domination in graphs. This was studied and introduced by E.J. Cockayne et.al [1, 4, 5]. Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible. The inverse domination in a graph was first found in the paper of Kulli [11] while Hedetmiemi et al. [9] introduced the concept of disjoint dominating sets in graphs. Moreover, for the general concepts not mentioned, readers may refer to [8].

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \varnothing$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v. The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{v \in X} N_G(v)$. The *closed neighborhood* of X in G is the set $N_G[X] = \bigcup_{v \in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ [resp. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$].

A subset S of $V(G)$ is a *dominating set* in G if for every $v \in V(G)\backslash S$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set in G. A dominating set S of $V(G)$ is a *secure dominating set* in G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set in G. The minimum cardinality of a secure dominating set in G, denoted by $\gamma_s(G)$, is called the *secure domination number* of G. A secure dominating set of cardinality $\gamma_s(G)$ is called a γ_s*-set* of G. Let D be a minimum dominating set in G. The dominating set $S \subseteq V(G) \setminus D$ is called an *inverse dominating set* with respect to D. The minimum cardinality of inverse dominating set is called an *inverse domination number* of G and is denoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is called γ^{-1}*-set* of G. In [9], Hedetniemi et al. defined the disjoint domination as $\gamma\gamma(G) = min\{|S_1| + |S_2| : S_1$ and S_2 are disjoint dominating sets of $G\}$. The two disjoint dominating sets whose union has cardinality $\gamma\gamma(G)$ is a $\gamma\gamma$*-pair* of

G.

The paper "Inverse secure domination in graphs", by Enriquez and Kiunisala [7], state that if D is a minimum secure dominating set in G, then a secure dominating set $S \subseteq (V(G) \setminus D)$ is called an *inverse secure dominating set* in G with respect to D. The *inverse secure domination number* of G denoted by $\gamma_s^{-1}(G)$ is the minimum cardinality of an inverse secure dominating set in G. An inverse secure dominating set of cardinality $\gamma_s^{-1}(G)$ is called γ_s^{-1}-*set*. Motivated by definition of inverse secure dominating set and disjoint disjoint dominating set, we define the following variant of domination in graphs. A disjoint secure dominating set in G is the set $C = D \cup S \subseteq V(G)$. The *disjoint secure domination number* of G denoted by $\gamma\gamma_s(G)$ is the minimum cardinality of a disjoint secure dominating set in G. A disjoint secure dominating set of cardinality $\gamma\gamma_s(G)$ is called $\gamma\gamma_s$-*set* of G. Unless otherwise stated, all graphs in this paper are assumed to be simple and connected.

2 Results

One of the classical result in the domination theory which was introduced by Ore in 1962 state the following theorem:

Theorem 2.1 [10] *Let G be a graph with no isolated vertex. If $S \subseteq V(G)$ is a γ-set, then $V(G) \setminus S$ is also a dominating set in G.*

Theorem 2.1 guarantees the existence of γ_s^{-1}-*set* and hence of $\gamma\gamma_s$-*set* in some graph G. Since $\gamma\gamma_s(G)$ does not always exists in a connected nontrivial graph G, we denote $\mathcal{SS}(G)$ a family of all graphs with disjoint restrained dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{SS}(G)$. From the definitions stated above, the following result is immediate.

Remark 2.2 *Let G be a connected graph of order $n \geq 3$. If D is a γ_s-set and S is a γ_s^{-1}-set of G, then $D \cap S = \varnothing$ and $C = D \cup S$ is a $\gamma\gamma_s$-set of G.*

Remark 2.3 *Let G be a connected graph of order $n \geq 3$. Then*

(*i*) $\gamma\gamma_s(G) \in \{2, 4, 5, 6, ..., n\}$,

(*ii*) $\gamma(G) \leq \gamma_s(G) \leq \gamma\gamma_s(G)$.

The next result says that the value of the parameter $\gamma\gamma_s$ ranges over all positive integers except 1 and 3.

Theorem 2.4 *Given positive integers k and n such that $n \geq 3$ and $k \in \{2, 4, 5, 6, ..., n\}$, there exists a connected nontrivial graph G with $|V(G)| = n$ and $\gamma\gamma_s(G) = k$.*

Proof. Consider the following cases:

*Case*1. Suppose $k = 2$.
Let $G = K_n$. Then, clearly, $|V(G)| = n$ and $\gamma\gamma_s(G) = 2$.

*Case*2. Suppose $4 \leq k < n$.
Let $P_2 = [x, y]$ and $P_{n-2} = [a_1, a_2, ..., a_{n-2}]$. Consider the graph $G = P_2 + P_{n-2}$ (see Figure 1).

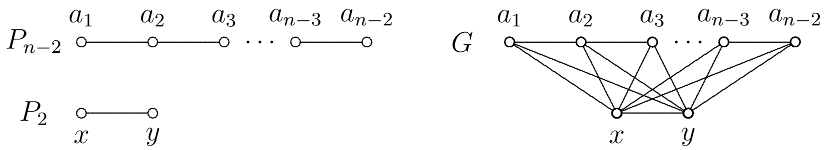

Figure 1: A graph G with $\gamma\gamma_s(G) = k$

Suppose that $n = 2i + 3$ for some positive integer i. If $i = 1$, then the set $D = \{x, a_1\}$ is a γ_s-*set* of G and $S = \{y, a_3\}$ is a γ_s^{-1}-*set* of G. Thus, $|V(G)| = 5 = n$ and $C = D \cup S$ is a $\gamma\gamma_s$-*set* of G, that is $\gamma\gamma_s(G) = 4 = k$. Similarly, if $i = 2$, then $|V(G)| = 7 = n$ and $C = \{x, y, a_2, a_4\}$ is a $\gamma\gamma_s$-*set* of G, that is $\gamma\gamma_s(G) = 4 = k$. If $i \geq 3$, then let $2k = n + 1$. The set $D = \{x, y\}$ is a γ_s-*set* of G and $S = \{a_{2i} : 1, 2, ..., \frac{n-3}{2}\}$ is a γ_s^{-1}-*set* of G. Thus, $C = D \cup S$ is a $\gamma\gamma_s$-*set* of G, that is $\gamma\gamma_s(G) = 2 + \frac{n-3}{2} = \frac{n+1}{2} = k$. Further, $|V(G)| = 2 + (n - 2) = n$.

Suppose that $n = 2i + 4$ for some positive integer i and $2k = n + 2$, then the set $D = \{x, y\}$ is a γ_s-*set* of G and $S = \{a_{2i} : 1, 2, ..., \frac{n-2}{2}\}$ is a γ_s^{-1}-*set* of G. Thus, $C = D \cup S$ is a $\gamma\gamma_s$-*set* of G, that is $\gamma\gamma_s(G) = 2 + \frac{n-2}{2} = \frac{n+2}{2} = k$. Further, $|V(G)| = 2 + (n - 2) = n$.

*Case*3. Suppose that $k = n$.
Let H be a connected graph of order m and $G = H \circ K_1$. Let $n = 2m$. Then $D = V(H)$ is a γ_s-*set* and $S = \bigcup_{x \in V(H)} V(K_1^x)$ is a γ_s^{-1}-*set* of G. Thus, $|V(G)| = |V(H \circ K_1)| = |V(H)|(|V(K_1)| + 1) = m(1 + 1) = n$ and $\gamma\gamma_s(G) = |D| + |S| = m + m = n = k$.

This proves the assertion. □

Corollary 2.5 *The difference $\gamma\gamma_s - \gamma_s$ can be made arbitrarily large.*

Proof: Let n be a positive integer. By Theorem 2.4, there exists a connected graph G such that $\gamma\gamma_s(G) = n + 2$ and $\gamma_s(G) = 2$. Thus, $\gamma\gamma_s(G) - \gamma_s(G) = n$. □

Since $\gamma_s(G)$ is the order of the minimum secure dominating set in G, it follows that $\gamma_s(G) \leq \gamma_s^{-1}(G)$. The following remark holds.

Remark 2.6 *Let G be a connected nontrivial graph of order $n \geq 3$. Then $\gamma_s(G) \leq \gamma_s^{-1}(G)$.*

Let G be a connected non-complete graph and let $\gamma_s^{-1}(G) = 2$. Since $\gamma_s(G) \leq \gamma_s^{-1}(G) = 2$ by Remark 2.6, it follows that either $\gamma_s(G) = 1$ or $\gamma_s(G) = 2$. If $\gamma_s(G) = 1$, then $G = K_n$. This is contrary to our assumption that G is non-complete. Therefore $\gamma_s(G) = 2$ and the following remarks follow.

Remark 2.7 *Let G be a connected non-complete graph of order $n \geq 4$. Then $\gamma\gamma_s(G) \neq 3$.*

Remark 2.8 *Let G be a connected non-complete graph of order $n \geq 4$. If $\gamma_s^{-1}(G) = 2$, then $\gamma_s(G) = 2$.*

Theorem 2.9 [2] *Let G be a graph of order $n \geq 1$. Then $\gamma_s(G) = 1$ if and only if $G = K_n$.*

Theorem 2.10 *Let G be a connected nontrivial graph of order $n \geq 2$. Then $\gamma\gamma_s(G) = 2$ if and only if $G = K_n$.*

Proof: Suppose that $\gamma\gamma_s(G) = 2$. Let $C = \{x, y\}$ be a $\gamma\gamma_s$-*set* of G. Then $C = D \cup S$ and $D \cap S = \varnothing$ where a nonempty set D is a γ_s-*set* of G. Let $D = \{x\}$. Then $\gamma_s(G) = 1$, that is, $G = K_n$ by Theorem 2.9.

For the converse, suppose that $G = K_n$. Let $D = \{x\}$ be a γ_s-*set* of G. Then $G - x$ is also a complete graph. Let $S = \{y\}$ be a γ_s-*set* of $G - x$ (and hence of G). Since $D \subseteq (V(G) \setminus S)$, it follows that S is a γ_s^{-1}-*set* of G with respect to D. Thus, $C = \{x, y\}$ is a $\gamma\gamma_s$-*set* of G. Hence, $\gamma\gamma_s^{-1}(G) = 2$. □

The following result is a quick consequence of Theorem 2.10.

Corollary 2.11 *Let G and H be two graphs. Then $\gamma\gamma_s(G + H) = 2$ if and only if G and H are complete graphs.*

We need the following results for our next characterization of the disjoint secure domination number of a graph G.

Theorem 2.12 [2] *Let G be a connected graph of order $n \geq 3$. Then $\gamma_s(G) = 2$ if and only if G is non-complete and there exists distinct vertices x and y that dominate G and satisfy one of the following conditions:*

(*i*) $N(x) \setminus \{y\} = N(y) \setminus \{x\} = V(G) \setminus \{x, y\}$.

(*ii*) $\langle N(x) \setminus N[y] \rangle$ *and* $\langle N(y) \setminus N[x] \rangle$ *are complete and for each* $u \in N(x) \cap N(y)$ *either* $\langle (N(x) \setminus N[y]) \cup \{u\} \rangle$ *or* $\langle (N(y) \setminus N[x]) \cup \{u\} \rangle$ *is complete.*

(*iii*) $N(x) \setminus \{y\} = V(G) \setminus \{x, y\}$, $N(x) \setminus N[y] \neq \varnothing$ *and* $\langle N(x) \setminus N[y] \rangle$ *is complete.*

The next result characterize the disjoint secure domination number equal to 2 of a graph G.

Theorem 2.13 *Let* G *be a connected non-complete graph of order* $n \geq 4$. *Then* $\gamma\gamma_s(G) = 4$ *if and only if there exists distinct vertices* x *and* y *that dominate* G *and satisfy one of the following conditions:*

(*i*) $N(x) \setminus \{y\} = N(y) \setminus \{x\} = V(G) \setminus \{x, y\} = V(H)$ *and* $\gamma_s(H) = 2$ *where* $xy \notin E(G)$ *whenever* $\gamma_s(H) = 1$.

(*ii*) $\langle N(x) \setminus N[y] \rangle$ *and* $\langle N(y) \setminus N[x] \rangle$ *are complete and for each* $u \in N(x) \cap N(y)$ *either* $\langle (N(x) \setminus N[y]) \cup \{u\} \rangle$ *or* $\langle (N(y) \setminus N[x]) \cup \{u\} \rangle$ *is complete.*

(*iii*) $N(x) \setminus \{y\} = V(G) \setminus \{x, y\}$, $N(x) \setminus N[y] \neq \varnothing$ *and* $\langle N(x) \setminus N[y] \rangle$ *is complete.*

Proof: Suppose that $\gamma\gamma_s(G) = 4$. Let $C = \{x, y, a, b\}$ be a $\gamma\gamma_s$-*set* of G. Since $\gamma_s(G) \leq \gamma_s^{-1}(G)$, it follows that $\gamma_s^{-1}(G) \neq 1$. Thus either $\gamma_s^{-1}(G) = 3$ or $\gamma_s^{-1}(G) = 2$. If $\gamma_s^{-1}(G) = 3$, then $\gamma_s(G) = 1$, that is, G is complete by Theorem 2.9. This is contrary to our assumption. Thus, $\gamma_s^{-1}(G) = 2$ and that $\gamma_s(G) = 2$ by Remark 2.8. In view of Theorem 2.12(i), $N(x) \setminus \{y\} = N(y) \setminus \{x\} = V(G) \setminus \{x, y\}$. Let $D = \{x, y\}$ be a γ_s-*set* of G and let $H = \langle V(G) \setminus D \rangle$. Since $\gamma_s^{-1}(G) = 2$, $S = \{a, b\}$ is a γ_s^{-1}-*set* of G. Thus $S \cap D = \varnothing$ and $S \subseteq (V(G) \setminus D) = V(H)$, that is, $\gamma_s(H) \leq |S| = 2$. Thus, either $\gamma_s(H) = 1$ or $\gamma_s(H) = 2$. If $\gamma_s(H) = 2$, then we have $N(x) \setminus \{y\} = N(y) \setminus \{x\} = V(G) \setminus \{x, y\} = V(H)$. If $\gamma_s(H) = 1$, then H is a complete graph. Since $G = \langle \{x, y\} \rangle + H$ is non-complete, $xy \notin E(G)$. This proves statement (i). Now, $\gamma_s(G) = 2$ implies that statement (ii) holds by Theorem 2.12(ii) or statement (iii) holds by Theorem 2.12(iii).

For the converse, suppose that there exists distinct vertices x and y that dominate G and statement (i), (ii), or (iii) holds.

Suppose first that statement (i) holds. Then $\gamma_s(G) = 2$ by Theorem 2.12. Let $D = \{x, y\}$ be a γ_s-*set* of G. Since $n \geq 4$, let $a, b \in (V(G) \setminus D) = V(H)$. If $\gamma_s(H) = 2$, then let $S = \{a, b\}$ be a γ_s-*set* of H and hence of G. Since $S \cap D = \varnothing$, $S \subseteq (V(G) \setminus D)$ is a γ_s^{-1}-*set* of G and hence $C = \{x, y, a, b\}$ is a $\gamma\gamma_s$-*set* of G. Thus $\gamma\gamma_s(G) = 4$. If $\gamma_s(H) = 1$ and $xy \notin E(G)$ then H is complete and $S = \{a, b\}$ is a secure dominating set in H and hence a γ_s^{-1}-*set* of G. Therefore $\gamma\gamma_s(G) = 4$.

Next, suppose that statement (ii) holds. Then $\gamma_s(G) = 2$ by Theorem 2.12. Let $D = \{x, y\}$ be a γ_s-*set* of G. Let $a \in N(x) \setminus N[y]$ and $b \in N(y) \setminus N[x]$ and let $S = \{a, b\}$. Then $ax, by \in E(G)$. Clearly, if $n = 4$, then S must be a γ_s^{-1}-*set* of G. Thus, $C = \{x, y, a, b\}$ is a $\gamma\gamma_s$-*set* of G, that is, $\gamma\gamma_s(G) = 4$. Suppose that $n > 4$. Let $z \in V(G)\backslash C$. If $z \in N(x)\backslash N[y]$, then $az \in E(G)$. If $z \in N(y)\backslash N[x]$, then $bz \in E(G)$. If $z \in N(x) \cap N(y)$, then $az \in E(G)$ or $bz \in E(G)$. Thus, for every $z \in V(G) \setminus S$, there exists $v \in S$ such that $zv \in E(G)$, that is, S is a dominating set in G. Now, $S_z = (S \setminus \{a\}) \cup \{z\} = \{b, z\}$ is dominating sets in G for all $z \in (N(x) \setminus N[y]) \cup \{u\}$ where $u \in N(x) \cap N(y)$ such that $\langle (N(x) \setminus N[u]) \cup \{u\} \rangle$ is complete. Similarly, $S'_z = (S \setminus \{b\}) \cup \{z'\} = \{a, z'\}$ is dominating sets in G for all $z' \in (N(y) \setminus N[x]) \cup \{u\}$ where $u \in N(x) \cap N(y)$ such that $\langle (N(y) \setminus N[x]) \cup \{u\} \rangle$ is complete. Consequently, S is a secure dominating set in G. Since $D \cap S = \varnothing$, $S \subseteq V(G) \setminus D$ is a γ_s^{-1}-*set* of G. Thus, $C = D \cup S$ is a $\gamma\gamma_s$-*set* of G and hence $\gamma\gamma_s(G) = 4$.

Suppose that statement (*iii*) holds. Then $\gamma_s(G) = 2$ by Theorem 2.12. Since $N(x) \setminus N[y] \neq \varnothing$ and $n \geq 4$, let $a, b \in N(x) \setminus N[y]$ and since $\langle N(x) \setminus N[y] \rangle$ is complete, $ab \in E(G)$. Let $D = \{a, y\}$ be a γ_s-*set* of G. Now, since G is connected and $N(x) \setminus \{y\} = V(G) \setminus \{x, y\} = (V(G) \setminus \{x\}) \setminus \{y\}$, it follows that $N(x) = V(G) \setminus \{x\}$, that is, $N[x] = N(x) \cup \{x\} = (V(G) \setminus \{x\}) \cup \{x\} = V(G)$. This implies that $\{x\}$ is a dominating set in G. Thus, $S = \{x, b\}$ is a dominating set in G. Since $S' = S \setminus \{b\} \cup \{a\} = \{x, a\}$ is a dominating set in G, it follows that S is a secure dominating set in G. Since $S \cap D = \varnothing$, it follows that $S \subseteq (V(G) \setminus D)$ is a γ_s^{-1}-*set* of G. Hence, $C = \{a, y, x, b\}$ is a $\gamma\gamma_s$-*set* of G, that is, $\gamma\gamma_s(G) = 4$. □

The following result is a direct consequence of Theorem 2.13.

Corollary 2.14 *Let G be a connected non-complete graph of order $n \geq 4$. Then $\gamma\gamma_s(G) = 4$ if $G = \langle \{x, y\} \rangle + (K_r \cup K_{n-r-2})$.*

The *join* of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Lemma 2.15 *Let G and H be connected non-complete graphs. If $D \subset V(G + H)$ is a γ_s-set, then there exists $S \subseteq V(G + H) \setminus D$ such that S is a secure dominating set in $G + H$.*

Proof: Suppose that $D \subset V(G + H)$ is a γ_s-*set*. If $D \cap V(H) = \varnothing$, then $D \subseteq V(G)$. Since H is a connected nontrivial graph, there exists $S \subseteq V(H)$ such that S is a secure dominating set in H and hence in $G + H$. Similarly, if $D \cap V(G) = \varnothing$, then there exists $S \subseteq V(G)$ such that S is a a secure dominating set in G and hence in $G + H$. Suppose that $D \cap V(G) \neq \varnothing$ and $D \cap V(H) \neq \varnothing$. Let $D_G = D \cap V(G)$ and $D_H = D \cap V(H)$. Then $D = D_G \cup D_H$ where $D_G \subset V(G)$ and $D_H \subset V(H)$. Since $G + H$ is a connected nontrivial graph, there exists $S \subseteq V(G+H)$ such that S is a secure dominating set in $G+H$. $\square$

The following result is the characterization of the disjoint secure dominating sets in the join of two graphs.

Theorem 2.16 *Let G and H be connected non-complete graphs. Then a subset C of $V(G+H)$ is a disjoint secure dominating set in $G+H$ if and only if $S' \subset C$ is a γ_s-set and*

(i) *$S' \subseteq V(G)$ where $|S'| \geq 2$, or*

(ii) *$S' \subseteq V(H)$ and $|S'| \geq 2$, or*

(iii) *$S' = S'_G \cup S'_H$ where $S'_G = \{v\} \subset V(G)$ and $S'_H = \{w\} \subset V(H)$ and*

- (a) *S'_G is a dominating set of G and S'_H is a dominating set of H; or*
- (b) *S'_G is dominating set of G and $(V(H) \setminus S'_H) \setminus N_H(S'_H)$ is a clique in H; or*
- (c) *S'_H is dominating set of H and $(V(G) \setminus S'_G) \setminus N_G(S'_G)$ is a clique in G; or*
- (d) *$(V(G) \setminus S'_G) \setminus N_G(S'_G)$ is a clique in G and $(V(H) \setminus S'_H) \setminus N_H(S'_H)$ is a clique in H, or*

(iv) *$S' = S'_G \cup S'_H$ where $S'_G \subseteq V(G)$ $(|S'_G| \geq 2)$ and $S'_H = \{w\} \subset V(H)$ and*

- (a) *S'_G is a dominating set, or*
- (b) *$(V(G) \setminus S'_G) \setminus N_G(S'_G)$ is a clique in G, or*

(v) *$S' = S'_G \cup G_H$ where $S'_G = \{v\} \subset V(G)$ and $S'_H \subseteq V(H)$ $(|S'_H| \geq 2)$ and*

- (a) *S'_H is a dominating set, or*
- (b) *$(V(H) \setminus S'_H) \setminus N_H(S'_H)$ is a clique in H, or*

(vi) *$S' = S'_G \cup S'_H$ where $S'_G \subseteq V(G)$ $(|S'_G| \geq 2)$ and $S'_H \subseteq V(H)$ $(|S'_H| \geq 2)$.*

Proof: Suppose that $C = D \cup S$ is a disjoint secure dominating set of $G+H$ where D is a γ_s-*set* and S is an inverse secure dominating set in $G+H$. Let $S' = D$. Then S' is a γ_s-*set* set in $G+H$.

Case1. Suppose that $S' \cap V(H) = \varnothing$. Then $S' \subseteq V(G)$. If $|S'| = 1$, then G is a complete graph (by Theorem 2.9) contrary to our assumption that G is non-complete. Thus, $|S'| \geq 2$. This proves statement (i). Similarly, if $S' \cap V(G) = \varnothing$ then $S' \subseteq V(H)$ and $|S'| \geq 2$ proving statement (ii).

Case2. Suppose that $S' \cap V(G) \neq \varnothing$ and $S' \cap V(H) \neq \varnothing$. Let $S'_G = S' \cap V(G)$ and $S'_H = S' \cap V(H)$. Then $S' = S'_G \cup S'_H$ where $S'_G \subseteq V(G)$ and $S'_H \subseteq V(H)$. Consider the following sub-cases.

Subcase1 Suppose that $S'_G = \{v\}$ and $S'_H = \{w\}$. If S'_G is a dominating set in G and S'_H is a dominating set in H, then we are done with statement $(iiia)$. Suppose that S'_G is a dominating set in G and S'_H is not a dominating set in H. Let $x \in (V(H) \setminus S'_H) \setminus N_H(S'_H)$. Since $S' = \{v, w\}$ is a secure dominating set in $G + H$ and $vx \in E(G + H)$, $S'_x = (S' \setminus \{v\}) \cup \{x\} = \{w, x\}$ is a dominating set in $G + H$ (and hence in H). Let $y \in V(H)$ such that $wy \notin E(H)$. Then $y \in V(H) \setminus S'_H$ and $y \notin N_H(S'_H)$. This implies that $y \in (V(H) \setminus S'_H) \setminus N_H(S'_H)$. Since $S'_x = \{w, x\}$ is a dominating set in H, $yx \in E(H)$. Suppose that there exists $z \in (V(H) \setminus S'_H) \setminus N_H(S'_H)$ such that $zy \notin E(H)$. Then $vz \in E(G + H)$ and $S'_y = (S' \setminus \{v\}) \cup \{y\} = \{w, y\}$ is not a dominating set in $G + H$ contrary to our assumption that S' is a secure dominating set in $G + H$. Thus, $zy \in E(H)$ for all $z \in (V(H) \setminus S'_H) \setminus N_H(S'_H)$. This implies that the sub-graph induced by $(V(H) \setminus S'_H) \setminus N_H(S'_H)$ is complete. Hence, $(V(H) \setminus S'_H) \setminus N_H(S'_H)$ is a clique in H. This proves statement $(iiib)$. Similarly, if S'_H is dominating set in H and S'_G is not a dominating set in G, then $(V(G) \setminus S'_G) \setminus N_G(S'_G)$ is a clique in G. This proves statement $(iiic)$. Furthermore, if S_G is not a dominating set in G and S'_H is not a dominating set in H, then statement $(iiid)$ holds.

Subcase2 Suppose that $|S'_G| \geq 2$ and $S'_H = \{w\}$. If S'_G is a dominating set, then (iva) holds. Suppose that S'_G is not a dominating set in G. Let $x \in (V(G) \setminus S'_G) \setminus N_G(S'_G)$. Then $xw \in E(G + H)$. Since $S' = S'_G \cup \{w\}$ is a secure dominating set in $G + H$, $S'_x = (S' \setminus \{w\}) \cup \{x\} = S'_G \cup \{x\}$ is a dominating set in $G + H$ (and hence in G). Let $y \in V(G)$ such that $vy \notin E(G)$ for all $v \in S'_G$. Then $y \in V(G) \setminus S'_G$ and $y \notin N_G(S'_G)$. This implies that $y \in (V(G) \setminus S'_G) \setminus N_G(S'_G)$. Since $S'_x = S'_G \cup \{x\}$ is a dominating set in G, $yx \in E(G)$. Suppose that there exists $z \in (V(G) \setminus S'_G) \setminus N_G(S'_G)$ such that $zy \notin E(G)$. Then $wz \in E(G + H)$ and $S'_y = (S' \setminus \{w\}) \cup \{y\} = S'_G \cup \{y\}$ is not a dominating set in $G + H$ contrary to our assumption that S' is a secure dominating set in $G + H$. Thus, $zy \in E(G)$ for all $z \in (V(G) \setminus S'_G) \setminus N_G(S'_G)$. This implies that the sub-graph induced by $(V(G) \setminus S'_G) \setminus N_G(S'_G)$ is complete. Hence, $(V(G) \setminus S'_G) \setminus N_G(S'_G)$ is a clique in G. This proves statement (ivb). Similarly, if $S'_G = \{v\}$ and $|S'_H| \geq 2$, then statement (va) or (vb) holds. Statement (vi) is clear.

For the converse, suppose that statement (i) holds. Let $D = S'$. Since $D \subset V(G + H)$ is a γ_s-*set*, there exists $S \subseteq V(G + H) \setminus D$ such that S is a secure dominating set in $G + H$ by Lemma 2.15. Thus, S is an inverse secure dominating set in $G + H$ with respect to D and hence $C = D \cup S$ is a disjoint secure dominating set in $G + H$. Similarly, if statement (ii) or (iii) or (iv) or (v) or (vi) holds, then in view of Lemma 2.15, $C = D \cup S$ is a disjoint secure dominating set in $G + H$. $\square$

Corollary 2.17 *Let G and H be connected non-complete graphs. If $\gamma_s(G) \geq 2$ or $\gamma_s(H) \geq 2$ then $4 \leq \gamma\gamma_s(G+H) \leq 8$.*

Proof: Let C be a nonempty subset of $V(G+H)$. Consider the following:

*Case*1. Suppose that $\gamma_s(G) = 2$ and $\gamma_s(H) = 2$.

Let $D = \{x, y\}$ be a γ_s-*set* set in G. Then $C \supset D$ is a disjoint secure dominating set in $G+H$ by Theorem 2.16(i). Let $S = \{x', y'\}$ be a γ_s-*set* in H. Then $S \subseteq V(G+H) \setminus D$ is a inverse secure dominating set in $G+H$. Thus $|C| = |D| + |S| = 4$. Since G and H are connected non-complete graphs, $G+H$ is non-complete. This implies that D and S are γ_s-*set* in $G+H$ and so $\gamma\gamma_s(G+H) = |D| + |S| = |C| = 4$.

*Case*2. Suppose that ($\gamma_s(G) = 2$ and $\gamma_s(H) \geq 3$) or ($\gamma_s(G) \geq 3$ and $\gamma_s(H) = 2$) or ($\gamma_s(G) = 3$ and $\gamma_s(H) = 3$).

Consider first that $\gamma_s(G) = 2$ and $\gamma_s(H) \geq 3$. Let $D = \{x, y\}$ be a γ_s-*set* set in G. Then $C \supset D$ is a disjoint secure dominating set in $G+H$ by Theorem 2.16(i). Moreover, D is a γ_s-*set* set in $G+H$. Let S be a secure dominating set in $G+H$. Suppose that S is a γ_s-*set* of H. Then $|S| \geq 3$ by assumption. Let S' be a minimum inverse secure dominating set of $G+H$ with respect to D. Then $|S'| \leq |S|$. If $S' = S$, then $C = D \cup S$ is a $\gamma\gamma_s$-*set* of $G+H$, that is, $\gamma\gamma_s(G+H) = |C| = |D| + |S| \geq 5$. If $S' < S$, then $S' \not\subseteq V(H)$, otherwise S is not a γ_s-*set* of H contrary to our assumption. Suppose that $S' \subset V(G) \setminus D$. Since G is non-complete, $|S'| \neq 1$ in view of Theorem 2.9. This implies that D and S' are both γ_s-*set* in G and hence γ_s-*set* in $G+H$. Since $D \cap S' = \varnothing$, $C = D \cup S'$ is a $\gamma\gamma_s$-*set* in $G+H$ and so $\gamma\gamma_s(G+H) = |C| = 4$. Suppose that $S' = S_G \cup S_H$ where $\varnothing \neq S_G \subseteq V(G) \setminus D$ and $\varnothing \neq S_H \subseteq V(H)$. If $|S_G| = 1 = |S_H|$, then $|S'| = 2$ and so $\gamma\gamma_s(G+H) = |C| = 4$. If either $|S_G| \geq 2$ or $|S_H| \geq 2$, then $|S'| \geq 3$ and so $\gamma\gamma_s(G+H) = |C| = |D| + |S| \geq 5$. Thus, $4 \leq \gamma\gamma_s(G+H)$. Similarly, if $\gamma_s(G) \geq 3$ and $\gamma_s(H) = 2$, then $4 \leq \gamma\gamma_s(G+H)$.

Next, if $\gamma_s(G) = 3$ and $\gamma_s(H) = 3$, then let $D = \{x, y, z\}$ be a γ_s-*set* in G and let $S = \{u, v, w\}$ be a γ_s-*set* in H. Let D' be a γ_s-*set* of $G+H$. If $D' \subset V(G)$, then $|D'| = |D|$ and hence D is a γ_s-*set* of $G+H$ and S is a γ_s^{-1}-*set* of $G+H$. Thus, $C = D \cup S$ is a $\gamma\gamma_s$-*set* of $G+H$ and so $4 < \gamma\gamma_s(G+H) = 6 < 8$. Similarly, if $D' \subseteq V(H)$, then $4 < \gamma\gamma_s(G+H) = 6 < 8$.

Suppose that $D' = D_G \cup D_H$ where $\varnothing \neq D_G \subseteq V(G)$ and $\varnothing \neq D_H \subseteq V(H)$. Consider that $|D_G| = 1$ and $|D_H| = 1$. If $D_G = \{x\}$ and $D_H = \{u\}$ are dominating sets in G and H respectively, then $D' = \{x, u\}$. Suppose there exists $y \in V(G) \setminus \{x\}$ such that $\{y\}$ is a dominating set in G. Then $\{x, y\}$ is a dominating set in G and for every $w \in V(G) \setminus \{x, y\}$, there exists $y' \in \{x, y\}$, say y, such that $yw \in E(G)$ and $(\{x, y\} \setminus \{y\}) \cup \{w\} = \{x, w\}$ is a dominating set in G. This implies that $\{x, y\}$ is a secure dominating set in G contrary to our assumption that $\gamma_s(G) = 3$. Thus, $\{y\} \subset V(G)$ is not a dominating set in G and so, $D_G = \{x\}$ is the only dominating set in G. Similarly, $D_H = \{u\}$ is the only dominating set in H. This implies that $S_G = \{x'\}$ $(x' \neq x)$ and $S_H = \{u'\}$ $(u' \neq u)$ are not dominating set in G and H respectively where $S' = S_G \cup S_H$, $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. Thus, $S' = \{x', u'\}$ is not a secure dominating set in $G + H$ for any $x' \in V(G) \setminus \{x\}$ and $u' \in V(H) \setminus \{u\}$. This implies that $|S'| \geq 3$ with $D' \cap S' \neq \varnothing$ and so $|C| = |D'| + |S'| \geq 5$. Thus, $4 < \gamma\gamma_s(G + H) = 5 < 8$.

*Case*3. Suppose that $\gamma_s(G) \geq 4$ and $\gamma_s(H) \geq 4$.

If $\gamma_s(G) = 4$ and $\gamma_s(H) = 4$, then let $D = \{x, y, z, z'\}$ be a γ_s-*set* of G and let $S = \{u, v, w, w'\}$ be a γ_s-*set* of H. Let D' be a γ_s-*set* of $G+H$. If $D' \subset V(G)$, then $|D'| = |D|$ and hence D is a γ_s-*set* of $G + H$ and S is a γ_s^{-1}-*set* of $G + H$. Thus, $C = D \cup S$ is a $\gamma\gamma_s$-*set* of $G + H$ and so $\gamma\gamma_s(G + H) = 8$. Similarly, if $D' \subseteq V(H)$, then $\gamma\gamma_s(G + H) = 8$. If $D' = D_G \cup D_H$ where $\varnothing \neq D_G \subseteq V(G)$ and $\varnothing \neq D_H \subseteq V(H)$, then it is clear that $4 < \gamma\gamma_s(G + H) < 8$ in view of similar arguments used in *Case*2.

Suppose that $\gamma_s(G) = 4$ and $\gamma_s(H) > 4$ (or $\gamma_s(G) > 4$ and $\gamma_s(H) \leq 4$). Let D' be a γ-*set* of G. If $|D'| = 1$, say $D' = \{x\}$, then let $D = D' \cup \{y, v\}$ where $y \in V(G) \setminus \{x\}$ and $v \in V(H)$. Since D' is a dominating set in G, it follows that D is a dominating set in $G + H$. Let $u \in V(G + H) \setminus D$. Then $ux \in E(G + H)$ and $D_u = D \setminus \{x\} \cup \{u\} = \{y, v, u\}$. Since $yw' \in E(G + H)$ for all $w' \in V(H)$ and $vw \in E(G + H)$ for all $w \in V(G)$, it follows that D_u is a dominating set in $G + H$. Thus, D is a secure dominating set in $G + H$. Let $z \in V(G) \setminus \{x, y\}$. Since G is non-complete, there exists $w \in V(G) \setminus \{x, y\}$ $(w \neq z)$ such that $wz, wy \notin E(G)$. Now, $D \setminus \{v\} = \{x, y\}$ is a dominating set in $G + H$, $xz \in E(G)$ and $\{x, y\}_z = (\{x, y\} \setminus \{x\}) \cup \{z\} = \{y, z\}$ is not a dominating set in G and so $\{x, y\}_z$ is not a dominating set in $G + H$. Thus, $D \setminus \{v\}$ is not a secure dominating set in $G+H$. Similarly, $D \setminus \{y\}$ and $D \setminus \{x\}$ are not secure dominating set in $G + H$. Thus, D must be a minimum secure dominating set in $G + H$, that is, D is a γ_s-*set* of $G + H$. Let S be a γ_s^{-1}-*set* of $G+H$. Then $3 = |D| \leq |S|$. Thus, $|C| = |D| + |S| \geq 6$, that is $\gamma\gamma_s(G+H) \geq 6$. Similar argument follows if $|D'| = 2$ or $|D'| = 3$ or $|D'| = 4$.

Suppose that $\gamma_s(G) > 4$ and $\gamma_s(H) > 4$. Let $D = \{x, y, u, v\}$ where $x, y \in V(G)$ and $u, v \in V(H)$. Then D is a dominating set in $G + H$. Let $z \in V(G + H) \setminus D$. If $z \in V(G)$, then $uz \in E(G+H)$ and $(D \setminus \{u\}) \cup \{z\}$ is a dominating set in $G+H$. If $z \in V(H)$, then $xz \in E(G+H)$ and $(D \setminus \{x\}) \cup \{z\}$ is a dominating set in $G + H$. Thus, D is a secure dominating set in $G + H$.

Let $D_x = D \setminus \{x\}$ (or $D_y = D \setminus \{y\}$) and $z \in V(H) \setminus \{u, v\}$. Clearly, D_x is a dominating set in $G + H$. Since H is non-complete, there exist $w \in V(H) \setminus \{u, v\}$ $(w \neq z)$ such that $wu, wv, wz \notin E(H)$. Now, $yz \in E(G+H)$ and $(D_x)_z = (D_x \setminus \{y\}) \cup \{z\} = \{u, v, z\}$ is not a dominating set in H and so $(D_x)_z$ is not a dominating set in $G + H$. Thus, D_x (or D_y)is not a secure dominating set in $G + H$. Similarly, if $D_v = D \setminus \{v\}$ (or $D_u = D \setminus \{u\}$)

and $z \in V(G) \setminus \{x, y\}$, then D_v (or D_u)is not a secure dominating set in $G + H$. Thus, D must be a minimum secure dominating set in $G + H$, that is, D is a γ_s-*set* of $G + H$. Let S be a γ_s^{-1}-*set* of $G + H$. Now, let $S = \{x', y', u', v'\}$ where $x', y' \in V(G) \setminus \{x, y\}$ and $u', v' \in V(H) \setminus \{u, v\}$. Then S is a minimum secure dominating set in $G + H$ by similar arguments above. Since $D \cap S = \varnothing$, $S \subseteq V(G + H) \setminus D$ is a minimum inverse secure dominating set in $G + H$ with respect to D. Thus, $4 = |D| \leq |S| = 4$, that is, $|C| = |D| + |S| = 8 \geq \gamma\gamma_s(G + H)$.□

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