On Some Results of a Torsion-Free Abelian Trace Group

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Abstract

In [6], given any torsion-free abelian groups G *and* H*, the pure trace of* H *in* G *is* $tr(H,G) = \left(\left\{\sum f(H) : f \in Hom(H,G)\right\}\right)_*$ *which is equivalent to the set* $\left\{g \in G : ng \in \left\langle \sum f(H), f \in Hom(H,G) \right\rangle \text{ for some } n \in Z^+\right\}$*. The pure trace* $tr(H,G)$ *is a pure fully invariant subgroup of G. A torsion-free abelian group* G *is a trace group if for every pure fully invariant subgroup* M *of* G*,* $M = tr(M,G)$*. In this paper, we give further results of pure trace, trace group and characterize the direct sum of the trace groups of a torsion-free abelian groups.*

Keywords: Trace group, pure trace, torsion-free abelian group, direct sum, pure fully invariant

1.0 Introduction

In [6], for any torsion-free abelian groups G and H, the pure trace of H in G is defined. Its algebraic and homological properties with respect to a short exact sequence are characterized and established. Meanwhile, in [6], the concept of pure trace of a torsion-free abelian group led to the notion of the trace group. The groups *Z*, *Q*, pure fully invariant-injective groups, completely decomposable groups, almost completely decomposable groups among others are found in [6] to be trace groups.

In this paper, we show that the irreducible and strongly irreducible groups are also trace groups. Moreover, we give more results on pure trace and characterizations concerning the direct sum of the trace group of a torsion-free abelian group.

Throughout this paper, all groups considered are torsion-free and abelian.

2.0 Preliminaries

We shall present some concepts and known results found in the indicated reference that are necessary for the main results of this paper.

Proposition 2.1 [4] If $f : G \to H$ is a homomorphism of groups and $N \lhd G$, $M \lhd H$ and $f(N) < M$, then f induces a homomorphism $\overline{f} : G/N \to H/M$ given by $aN \mapsto f(a)M$.

Definition 2.1 [3] A subgroup *M* of a group G is ***fully invariant*** in G if $f(M) < M$ for all $f \in End(G)$.

Theorem 2.1 [2] A fully invariant subgroup of a fully invariant subgroup of A is fully invariant in A.

Definition 2.2 [2] Let $\{B_i : i \in I\}$ be a family of subgroups of *A*. Then the set

$$\sum_{i \in I} B_i = \{b_1 + b_2 + \cdots + b_n : b_i \in B_i, \text{ and } n \in Z^+\}.$$

Definition 2.3 [2] Let $\{B_i : i \in I\}$ be a family of subgroups of A such that the following conditions hold;

i. $\sum_{i \in I} B_i = A$

ii. for every $i \in I$, $B_i \cap \sum_{j \in I, j \neq i} B_j = \{0\}$.

Then *A* is said to be the ***direct sum*** of its subgroups B_i, denoted by $A = \oplus_{i \in I} B_i$ or $A = B_1 \oplus B_2 \oplus \cdots \oplus B_n$, if $I = \{1, 2, \cdots, n\}$.

Theorem 2.2 [2] If $G = A \oplus B$, then every $f \in End(A)$ extends to an endomorphism of G by assigning the image of the second component of elements of G to 0.

Lemma 2.1 [2] If $G = B \oplus C$ and A is a fully invariant subgroup of G, then $A = (A \cap B) \oplus (A \cap C)$.

Definition 2.4 [4] Let $\{B_i : i \in I\}$ be a set of groups. An i^{th} tuple $f = (\cdots, b_i, \cdots)$ over this collection of groups B_i has exactly one coordinate b_i for each $i \in I$, that is, if f is a function defined over I, then $f(i) = b_i \in B_i$ for every $i \in I$. The set of all i^{th} tuples is called the ***direct product*** of the groups B_i, denoted by $\prod_{i \in I} B_i$.

Theorem 2.3 [2] Let $A, B, A_i\ (i \in I)$ and $B_j\ (j \in J)$ be abelian groups. Then

i. $Hom(\oplus_{i \in I} A_i, B) \cong \prod_{i \in I} (A_i, B)$;

ii. $Hom(A, \prod_{j \in J} B_j) \cong \prod_{j \in J} Hom(A, B_j)$.

Remark: 2.3.1 [4] If I is finite, then $\sum_{i \in I} A_i = \prod_{i \in I} A_i$.

Definition 2.5 [2] An abelian group G is said to be ***torsion*** if every element of G has finite order.

Example 2.5.1 Let A be an abelian group and let $T = \{x \in A : |x| < \infty\}$. Then T is a subgroup of A called the torsion group of A.

Definition 2.6 [2] An abelian group G is ***torsion-free*** if every nonzero element of G has infinite order.

Example 2.6.1 The groups Z and Q are torsion-free since $nx = 0$, only if $x = 0$ for all $n \in Z^+$, $\forall x \in Z$ or $\forall x \in Q$.

Definition 2.7 [2] A group G is divisible if $G = nG$ for all $0 \neq n \in Z^{+}$.

Example 2.7.1 The groups $\{0\}$ and Q are divisible since $\{0\} = n\{0\}$ and $Q = nQ$ for all $n \in Z^{+}$.

Definition 2.8 [2] A subgroup M of a group G is said to be ***pure*** in G, denoted by $M \leq_{*} G$, if whenever $m = ny \in M$ for some $n \in Z^{+}$, $y \in G$, then there exists $m_1 \in M$ such that $m = nm_1$. Equivalently, M is pure in G if and only if $M \cap nG = nM$ for all $n \in Z^{+}$.

Example 2.8.1 Consider the group Z. Then $\{0\}$ and Z are the only pure subgroups of Z.

Proof: For all $n \in Z^{+}$, $n\{0\} = \{0\} = nZ \cap \{0\}$ and $nZ = nZ \cap Z$. Hence, $\{0\}$ and Z are pure subgroups of Z. Consider $mZ \leq Z$ for some $m \in Z^{+}$. Then for all $n \in Z^{+}$, $n(mZ) = nZ \cap mZ$ if $\gcd(n,m) = 1$. But for all $n, m \in Z^{+}$, $\gcd(n,m)$ is not always equal to 1. Thus $\{0\}$ and Z are the only pure subgroups of Z. ■

Example 2.8.2 The group of rational numbers Q has no proper nontrivial pure subgroups.

Proof: Let $\{0\} \neq H \leq_{*} Q$. Then $H \cap nQ = nH$ for all $n \in Z^{+}$. Since Q is divisible, $Q = nQ$ for all $n \in Z^{+}$. Hence, $H \cap nQ = H \cap Q = H$. Thus, $H = nH$, that is, H is divisible, by Definition 2.7. Let $\frac{a}{b} \in H$, $a, b \in Z - \{0\}$. Then $bH = H$ and so $a = b\left(\frac{a}{b}\right) \in H = aH$. Thus, there exists $h \in H$ such that $a = ah$ for some $h \in H$. Hence, $h = 1 \in H$. Since $H = nH$, $\frac{1}{n} = \left(\frac{1}{n}\right)n \in H = nH$ for all $0 \neq n \in Z$. This implies that $\frac{1}{n} \in H$ since H is torsion – free. Therefore, $H = Q$. ■

Definition 2.9 [2] If G is torsion-free and $X \subseteq G$, then

$$(X)_{*} = \{g \in G : ng \in \langle X \rangle \text{ for some } n \in Z^{+}\}$$

is called the ***pure subgroup of G generated by*** X.

Theorem 2.4 [2] Let B, C be subgroups of A such that $C \leq B \leq A$. Then the following hold.

i. If C is pure in B and B is pure in A, then C is pure in A.
ii. If B is pure in A, then B/C is pure in A/C.
iii. If C is pure in A, B/C pure in A/C, then B is pure in A.

Definition 2.10 [2] A subgroup B of A is a ***direct summand*** of A if $A = B \oplus C$ for some $C \leq A$.

Theorem 2.5 [2] Every direct summand is a pure subgroup and in torsion-free groups, the intersection of pure subgroups is again pure.

Definition 2.11 [1] A group G is said to be ***irreducible*** if it has no proper nontrivial pure fully invariant subgroups.

Example 2.11.1 The groups Z and Q are irreducible groups since it contains no proper nontrivial pure fully invariant subgroups.

Definition 2.12 [5] For any groups G and H, we say that

i. G is quasi-contained in H $\left(G \dot{\subseteq} H\right)$ if $nG \subseteq H$ for some $0 \neq n \in Z^{+}$;

ii. G is quasi – equal to H $\left(G \dot{=} H\right)$ if $G \dot{\subseteq} H$ and $H \dot{\subseteq} G$.

Definition 2.13 [5] A group G is said to be strongly irreducible if G is quasi-equal to each of its nontrivial fully invariant subgroups.

Remark 2.13.1 A strongly irreducible group is irreducible.

Proof: If *H* is a nontrivial fully invariant subgroup of a strongly irreducible group *G*, then $nG \subseteq H$ for some positive integer n. If in addition, $H \leq_{*} G$, then $nH = nG \bigcap H = nG$ for all $n \in Z^{+}$. Since *H* is torsion – free, $H = G$. Thus, a strongly irreducible group is irreducible. ■

The following notions and results deals with the pure trace and trace group of a torsion- free abelian group.

Definition 2.14 [6] For any groups G and H, the pure trace of H in G is

$$tr(H,G) = \left(\left\{\sum f(H) : f \in Hom(H,G)\right\}\right)_{*}$$
$$= \left\{g \in G; ng \in \langle f(H), f \in Hom(H,G)\rangle \text{ for some } n \in Z^{+}\right\}$$

Example 2.14.1 Consider the group of integers Z. Then by Definition 2.14, $tr(Z,Z) \subseteq Z$. Let $z \in Z$. Observe that the identity map $i_z \in End(Z)$ and $z = i_z(z) \in \sum f(Z)$, $f \in End(Z)$. So taking $n = 1$, $nz \in \left\langle \sum f(Z) \right\rangle$, $f \in End(Z)$. Thus, $z \in tr(Z,Z)$ and $Z \subseteq tr(Z,Z)$. Hence, $Z = tr(Z,Z)$.

Also, the pure trace of $\{0\}$ in Z is

$$tr(\{0\},Z) = \left\{y \in Z : ny \in \sum f(\{0\}), f \in Hom(\{0\},Z) \exists n \in Z^{+}\right\}$$
$$= \left\{y \in Z : ny \in \sum \{0\} \text{ for some } n \in Z^{+}\right\}$$
$$= \left\{y \in Z : ny = 0 \text{ for some } n \in Z^{+}\right\}$$
$$= \left\{y \in Z : y = 0\right\} \text{ since } n \in Z^{+}$$
$$= \{0\}.$$

Example 2.14.2 Consider the rational group Q. Then by Definition 2.14, $tr(Q,Q) \subseteq Q$. Let $q \in Q$. Observe that the identity map $i_Q \in End(Q)$ and $q = i_Q(q) \in \sum f(Q)$, $f \in End(Q)$. So taking $n = 1$, $nq \in \langle \sum f(Q) \rangle$, $f \in End(Q)$. Thus, $q \in tr(Q,Q)$ and $Q \subseteq tr(Q,Q)$. Thus, $Q = tr(Q,Q)$.

Also, the pure trace of $\{0\}$ in Q is

$$\begin{aligned} tr(\{0\},Q) &= \{q \in Q : nq \in \sum f(\{0\}),\ f \in Hom(\{0\},Q) \exists n \in Z^+\} \\ &= \{q \in Q : nq \in \sum \{0\} \text{ for some } n \in Z^+\} \\ &= \{q \in Q : nq = 0 \text{ for some } n \in Z^+\} \\ &= \{q \in Q : q = 0\} \text{ since } n \in Z^+ \\ &= \{0\}. \end{aligned}$$

The preceding examples can be generalized as follows.

Example 2.14.3 Let *G* be any torsion-free abelian group. Then by Definition 2.14, $tr(G,G) \subseteq G$. Let $g \in G$. Note that the identity map $i_G \in End(G)$ and $g = i_G(g) \in \sum f(G)$, $f \in End(G)$. Taking $n = 1$, $ng \in \langle \sum f(G) \rangle$, $f \in End(G)$. Hence, $g \in tr(G,G)$ and $G \subseteq tr(G,G)$. Thus, $G = tr(G,G)$.

Also, the pure trace of $\{0\}$ in G is

$$\begin{aligned} tr(\{0\},G) &= \{g \in G : ng \in \sum f(\{0\}),\ f \in Hom(\{0\},G) \exists n \in Z^+\} \\ &= \{g \in G : ng \in \sum \{0\} \text{ for some } n \in Z^+\} \\ &= \{g \in G : ng = 0 \text{ for some } n \in Z^+\} \\ &= \{g \in G : g = 0\} \text{ since } n \in Z^+ \\ &= \{0\}. \end{aligned}$$

Proposition 2.2 For any torsion-free abelian groups G and H, $tr(H,G)$ is a pure fully invariant subgroup of G.

Proof: Let G and H be any torsion-free abelian groups. Since $\sum f(H) \subset G$, $f \in Hom(H,G)$, it follows by Definition 2.14 and Definition 2.9 that $tr(H,G) \leq_* G$. Let $\alpha \in End(G)$ and $g \in tr(H,G)$. Then for some $n \in Z^+$, for some finite number of $h_i \in H$ and $f_i \in Hom(H,G)$, $ng = \sum f_i(h_i)$. Thus,

$$\begin{aligned} n(\alpha(g)) &= \alpha(ng) \\ &= \alpha\left(\sum f_i(h_i)\right) \\ &= \sum \alpha f_i(h_i), \text{ where } \alpha f_i \in Hom(H,G). \end{aligned}$$

This implies that $n(\alpha(g)) \in \sum \alpha f_i(H)$. Hence, $\alpha(g) \in tr(H,G)$ and $tr(H,G)$ is fully invariant in G. Therefore, $tr(H,G)$ is a pure fully invariant subgroup of G. ■

Lemma 2.2 [6] Let *G* be any torsion-free abelian group. If M is a subgroup of G, then $M \subseteq tr(M,G)$ with equality if and only if M is pure and every $f \in Hom(M,G)$ maps M into M.

Definition 2.15 [6] A torsion-free abelian group G is a ***trace group*** if for every pure fully invariant subgroup M of G, $M = tr(M,G)$.

Example 2.15.1 The groups Z and Q are trace groups since their only pure fully invariant subgroups are the trivial ones and itself with $tr(Z,Z) = Z$, $tr(\{0\},Z) = \{0\}$ and $tr(Q,Q) = Q$ and $tr(\{0\},Q) = \{0\}$.

3.0 Main Results

We now give and prove some results and characterizations pertaining to the direct sum of pure trace and trace groups of torsion-free abelian groups.

Proposition 3.1 If G is irreducible, then G is a trace group.

Proof: Suppose G is irreducible. Then by Definition 2.11, $\{0\}$ and G are the only pure fully invariant subgroups of G. But $tr(\{0\},G) = \{0\}$ and $tr(G,G) = G$, by Example 2.14.3. Thus, G is a trace group. ■

Proposition 3.2 A strongly irreducible group is a trace group.

Proof: A strongly irreducible group G is irreducible, by Remark 2.13.1. Hence, by Proposition 3.1, G is a trace group. ■

The following are some properties of a pure trace of a torsion-free abelian group.

Lemma 3.1 For any torsion-free abelian groups *H*, *A*, and *B*,

$$tr(H, A \oplus B) = tr(H, A) \oplus tr(H, B).$$

Proof: Let $x \in tr(H, A \oplus B)$. Then for some $n \in Z^+$, some finite number of $b_i \in H$, and $f_i \in Hom(H, A \oplus B)$, $nx = \sum f_i(b_i)$. By Theorem 2.3,

$$Hom(H, A \oplus B) = Hom(H, A) \oplus Hom(H, B).$$

So, $f_i = f_{i_1} + f_{i_2}$, $f_{i_1} \in Hom(H, A)$ and $f_{i_2} \in Hom(H, B)$ Thus,

$$nx = \sum \left(f_{i_1} + f_{i_2}\right)(b_i)$$

$$= \sum \left(f_{i_1}(b_i) + f_{i_2}(b_i)\right)$$

$$= \sum f_{i_1}(b_i) + \sum f_{i_2}(b_i).$$

Hence, $x \in tr(H,A) \oplus tr(H,B)$ that is,

$$tr(H, A \oplus B) \subseteq tr(H,A) \oplus tr(H,B).$$

Let $y \in tr(H,A) \oplus tr(H,B)$. Then, $y = y_1 + y_2$ where $y_1 \in tr(H,A)$ and $y_2 \in tr(H,B)$. Hence, for some $m_1, m_2 \in Z^+$, some finite number of $f_{i_1} \in Hom(H,A)$, $f_{i_2} \in Hom(H,B)$ and $b_{i_1}, b_{i_2} \in H$, $m_1 y_1 = \sum_i f_{i_1}(b_{i_1})$, $m_2 y_2 = \sum_i f_{i_2}(b_{i_2})$. Thus,

$$m_1 m_2 y = m_2(m_1 y_1) + m_1(m_2 y_2)$$

$$= m_2 \sum f_{i_1}(b_{i_1}) + m_1 \sum f_{i_2}(b_{i_2})$$

$$= \sum m_2 f_{i_1}(b_{i_1}) + \sum m_1 f_{i_2}(b_{i_2})$$

where $m_2 f_{i_1} \in Hom(H,A)$ and $m_1 f_{i_2} \in Hom(H,B)$. Since $A \subseteq A \oplus B$ and $B \subseteq A \oplus B$, $m_2 f_{i_1}, m_1 f_{i_2} \in Hom(H, A \oplus B)$. Thus, $y \in tr(H, A \oplus B)$ and

$$tr(H,A) \oplus tr(H,B) \subseteq tr(H, A \oplus B).$$

Therefore,

$$tr(H, A \oplus B) = tr(H,A) \oplus tr(H,B). \blacksquare$$

Lemma 3.2 For any torsion-free abelian groups *A*, *B*, *C* and *D*, where *A* is fully invariant in *C* and *B* is fully invariant in *D*,

$$tr(A \oplus B, C \oplus D) = tr(A,C) \oplus tr(B,D).$$

Proof: Let $x \in tr(A \oplus B, C \oplus D)$. Then for some $n \in Z^+$, some finite number of $f_i \in Hom(A \oplus B, C \oplus D)$, $a_i + b_i \in A \oplus B$, $nx = \sum f_i(a_i + b_i)$. Since A is fully invariant in *C* and *B* is fully invariant in *D*, it follows that $Hom(B,C) = \{0\}$ and $Hom(A,D) = \{0\}$. Thus, by Theorem 2.3,

$$Hom(A \oplus B, C \oplus D) = Hom(A \oplus B, C) \oplus Hom(A \oplus B, D)$$

$$= Hom(A,C) \oplus Hom(B,C) \oplus Hom(A,D) \oplus Hom(B,D)$$

$$= Hom(A,C) \oplus \{0\} \oplus \{0\} \oplus Hom(B,D)$$

$$= Hom(A,C) \oplus Hom(B,D).$$

Hence, $f_i = f_{i_1} + f_{i_2}$ with $f_{i_1} \in Hom(A,C)$ and $f_{i_2} \in Hom(B,D)$. Thus,

$$\begin{aligned} nx &= \sum \left(f_{i_1} + f_{i_2}\right)(a_i + b_i) \\ &= \sum f_{i_1}(a_i) + \{0\} + \{0\} + \sum f_{i_2}(b_i) \\ &= \sum f_{i_1}(a_i) + \sum f_{i_2}(b_i). \end{aligned}$$

So, $x \in tr(A,C) \oplus tr(B,D)$ and $tr(A \oplus B, C \oplus D) \subseteq tr(A,C) \oplus tr(B,D)$. Conversely, suppose $y \in tr(A,C) \oplus tr(B,D)$. Then

$$y = y_1 + y_2,\ y_1 \in tr(A,C) \text{ and } y_2 \in tr(B,D).$$

Hence, for some $n_1, n_2 \in Z$, some finite number of $g_{j_1} \in Hom(A,C)$, $g_{j_2} \in Hom(B,D)$ and $a_j \in A$, $b_j \in B$, $n_1 y_1 = \sum_j g_{j_1}(a_j)$ and $n_2 y_2 = \sum_j g_{j_2}(b_j)$. Thus,

$$\begin{aligned} n_1 n_2 y &= n_2(n_1 y_1) + n_1(n_2 y_2) \\ &= n_2 \sum g_{j_1}(a_j) + n_1 \sum g_{j_2}(b_j) \\ &= \sum n_2 g_{j_1}(a_j) + \sum n_1 g_{j_2}(b_j) \end{aligned}$$

where $n_2 g_{j_1} \in Hom(A,C)$ and $n_1 g_{j_2} \in Hom(B,D)$. Since $A, B \subset A \oplus B$, and $C, D \subset C \oplus D$, $n_2 g_{j_1}, n_1 g_{j_2} \in Hom(A \oplus B, C \oplus D)$. Hence,

$$tr(A,C) \oplus tr(B,D) \subseteq tr(A \oplus B, C \oplus D).$$

And so, $tr(A \oplus B, C \oplus D) = tr(A,C) \oplus tr(B,D)$. ■

Lemma 3.3 Let *N* and *M* be subgroups of a torsion-free abelian group *G* such that $M < N$ and $M \leq_* G$. If $tr(N/M, G/M) = N/M$, then $tr(N,G) = N$.

Proof: Suppose $tr(N/M, G/M) = N/M$. Let $x \in tr(N,G)$. Then for some $s \in Z^+$, some finite sets $f_i \in Hom(N,G)$ and $n_i \in N$, $sx = \sum f_i(n_i)$. Now, by Proposition 2.1, $f_i \in Hom(N,G)$ induces $\overline{f_i} \in Hom(N/M, G/M)$ given by $\overline{f_i}(n_i + M) = f_i(n_i) + M$. Hence, $\sum \overline{f_i}(n_i + M) \in tr(N/M, G/M) = N/M$, by hypothesis. Note that

$$\sum \bar{f}(n_i + M) = \sum (f_i(n_i) + M)$$
$$= \left(\sum f_i(n_i)\right) + M$$
$$= tx + M.$$

Thus, $sx + M \in N/M$, say $sx + M = n + M$ for some $n \in N$. Hence, $sx - n \in M$ which implies that $sx - n = m$ for some $m \in M$. So, $sx = m + n \in M + N \leq N$ since $M \leq N$. Now, $N \leq_* G$, by Theorem 2.4 (iii). Hence,

$$sx \in N \cap sG = sN \text{ implies } x \in N.$$

Thus, $tr(N, G) \subseteq N$. But $N \subseteq tr(N, G)$ by Lemma 2.2. Therefore, $tr(N, G) = N$. ■

The succeeding results characterize the trace group with respect to its direct sum.

Theorem 3.1 Let G be a trace group. If $G = A \oplus B$ and A is fully invariant in G then *A* is a trace group.

Proof: Suppose $G = A \oplus B$ such that G is a trace group and *A* is fully invariant in G. Let M be a pure fully invariant subgroup of *A*. Since $A \leq_* G$, by Theorem 2.4(i), M is also pure in G. Since M is fully invariant in A and A is fully invariant in G, by Theorem 2.1, M is fully invariant in G. Hence, by Definition 2.15 and by Lemma 3.1,

$$M = tr(M, G)$$
$$= tr(M, A \oplus B)$$
$$= tr(M, A) \oplus tr(M, B).$$

Claim: $tr(M, B) = \{0\}$.

Since M is fully invariant in A, that is, for all $m \in M$, $f \in End(A)$, $f(m) \in M \subset A$ and $A \cap B = \{0\}$, it follows that $Hom(M, B) = \{0\}$. Hence, $\sum f_i(m_i) = 0$, for all $m_i \in M$ where $f_i \in Hom(M, B)$. So, $tr(M, B) = \{0\}$. Therefore, $M = tr(M, A)$ and A is a trace group. ■

The following Lemma will be utilized to prove the succeeding Theorem 3.2.

Lemma 3.4 Let $G = A \oplus B$ where $A, B \leq_* G$ and let M be a pure subgroup of G. Then $M \cap A$ and $M \cap B$ are pure in A and in B, respectively.

Proof: Clearly, $n(M \cap A) \subseteq nA \cap (M \cap A)$ for all $n \in Z^+$. Let $x \in nA \cap (M \cap A)$. Then, $x \in nA$ and $x \in M \cap A$. So, $x = na$ for some $a \in A$, $x \in M$, and $x \in A$. Since $x \in nG$, $x \in nG \cap M = nM$. Thus, $x = na \in nM$ or $a \in M$ since M is torsion – free. Hence, $a \in M \cap A$ and so $x = na \in n(M \cap A)$ which implies that $nA \cap (M \cap A) \subseteq n(M \cap A)$ for all $n \in Z^+$. Therefore, $n(M \cap A) = nA \cap (M \cap A)$ and $M \cap A \leq_* A$. Similarly, $M \cap B \leq_* B$. ■

If $G = A \oplus B,$ then Theorem 2.2 implies that $End(A) \subseteq End(G)$ and $End(B) \subset End(G).$ This fact will be used to show that a direct sum of trace group is a trace group.

Theorem 3.2 If $G = A \oplus B,$ where A and B are trace groups, then G is a trace group.

Proof: Suppose $G = A \oplus B$ where A and B are trace groups. Then by Theorem 2.5, $A, B \leq_* G.$ Let M be pure fully invariant subgroup of G. Then

$$M = (M \cap A) \oplus (M \cap B) \text{ by Lemma 2.1.}$$

By Theorem 2.5, $M \cap A$ and $M \cap B$ are pure in G and by Lemma 3.4, $M \cap A \leq_* A$ and $M \cap B \leq_* B.$ Let $f \in End(A) \subset End(G)$ and $g \in End(B) \subset End(G).$ Then, $f(M \cap A) \subseteq M \cap A$ and $g(M \cap B) \subseteq M \cap B$ since M is fully invariant in G. Thus, $M \cap A$ and $M \cap B$ are fully invariant in A and B, respectively. Since A and B are trace groups, it follows by Definition 2.15 and Lemma 3.2 that

$$\begin{aligned} M &= (M \cap A) \oplus (M \cap B) \\ &= tr(M \cap A, A) \oplus tr(M \cap B, B) \\ &= tr((M \cap A) \oplus (M \cap B), A \oplus B). \end{aligned}$$

Therefore, G is a trace group. ■

Corollary 3.1 Let G be a group and $\{G_1, G_2, \cdots, G_n\}$ a collection of trace groups. If $G = \oplus_{i=1}^{n} G_i$, then *G* is a trace group.

Proof: Suppose $G = \oplus_{i=1}^{n} G_i$. Then we proceed by induction on *n*. If *n* = 2, then $G = G_1 \oplus G_2$ and by Theorem 3.2, *G* is a trace group. Now, suppose for *k* > 2, $G = \oplus_{i=1}^{k} G_i$ is a trace group. Then we wish to show that $G = \oplus_{i=1}^{k+1} G_i$ is a trace group. Since $G = \oplus_{i=1}^{k} G_i \oplus G_{k+1}$ with $\oplus_{i=1}^{k} G_i$ and G_{k+1} are trace groups, it follows from Theorem 3.2 that $G = \oplus_{i=1}^{k+1} G_i$ is a trace group. So, by Principle of Mathematical Induction (*PMI*), *G* is a trace group. ■

The following Corollary follows from Theorem 3.1 and Corollary 3.1.

Corollary 3.2 Let $G = \oplus G_i$, G_i fully invariant in G for all i. Then G is a trace group if and only if G_i is a trace group.

Proof: Suppose G is a trace group. Since $G = \oplus G_i$, and each G_i is fully invariant in G, by Theorem 3.1, each G_i is a trace group. Conversely, if each G_i is a trace group, then by Corollary 3.1, $\oplus G_i$ is a trace group. Therefore, it follows that G is a trace group. ■

4.0 Conclusions

In this paper, the irreducible and strongly irreducible groups were found to be trace groups. Moreover, the concepts of pure trace and trace groups also hold on the direct sum of pure trace and trace groups of torsion-free abelian groups. More generally, we have shown that if *G* is a torsion-free abelian group and $\{G_1, G_2, \cdots, G_m\}$ a collection of trace groups where $G = \oplus_{i=1}^{m} G_i$, then *G* is a trace group. Furthermore, if $G = \oplus G_i$, where G_i fully invariant in a torsion-free abelian group G for all i, then G is a trace group if and only if G_i is a trace group.

5.0 References

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