On Some Results of a Torsion-Free Abelian Kernel Group

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Abstract

In [6], for any torsion-free abelian groups G *and* H*, the kernel of* H *in* G *is* $\ker(G,H)=\bigcap_{f\in Hom(H,G)}\ker f$*. The kernel of* H *in* G *is a pure fully invariant subgroup of G. A torsion-free abelian group* G *is a kernel group if* $M=\ker(G,G/M)$ *for every pure fully invariant subgroup* M *of* G*. This paper shall give further results and characterizations of the direct sum of a kernel and kernel groups of a torsion-free abelian group.*

Keywords: Kernel, kernel group, torsion-free abelian group, direct sum, pure fully invariant

1.0 Introduction

In [6], given any torsion-free abelian groups G and H, the kernel of H in G is defined. Its algebraic and homological properties with respect to a short exact sequence are given. Moreover, in [6], the notion of the kernel of a torsion-free abelian group gives rise to the concept of the kernel group. The groups *Z*, *Q*, pure fully invariant-projective groups, almost completely decomposable groups and completely decomposable groups are some of the groups shown in [6] to be kernel group.

In this paper, we prove that the irreducible and the strongly irreducible groups are likewise kernel groups. Also, we characterize the direct sum of a kernel and kernel group of a torsion-free abelian group.

Throughout this paper, all groups considered are torsion-free and abelian.

2.0 Preliminary Concepts and Known Results

We shall recall some concepts and known results found in the indicated reference that we need for the main results of this paper.

Proposition 2.1 [4] If G is a group and $\{M_i : i\in I\}$ is a nonempty family of subgroups of G, then $\bigcap_{i\in I} M_i$ is a subgroup of G.

Theorem 2.1 [4] If $f:G\to H$ is a homomorphism of groups, then $\ker f \lhd G$. Conversely, if $N\lhd G$, the map $\pi:G\to G/N$ given by $\pi(a)=aN$ is an epimorphism with kernel N called the canonical epimorphism or projection.

Definition 2.1 [3] A subgroup *M* of a group G is ***fully invariant*** in G if $f(M) < M$ for all $f \in End(G)$.

Theorem 2.2 [2] A fully invariant subgroup of a fully invariant subgroup of A is fully invariant in A.

Definition 2.2 [2] Let $\{B_i : i \in I\}$ be a family of subgroups of *A*. Then the set

$$\sum_{i\in I} B_i = \{b_1 + b_2 + \cdots + b_n : b_i \in B_i, \text{and } n \in Z^+\}.$$

Definition 2.3 [2] Let $\{B_i : i \in I\}$ be a family of subgroups of A such that the following conditions hold;

i. $\sum_{i\in I} B_i = A$

ii. for every $i \in I$, $B_i \cap \sum_{j\in I, j\neq i} B_j = \{0\}$.

Then *A* is said to be the ***direct sum*** of its subgroups B_i, denoted by $A = \oplus_{i\in I} B_i$ or $A = B_1 \oplus B_2 \oplus \cdots \oplus B_n$, if $I = \{1, 2, \cdots, n\}$.

Theorem 2.3 [2] If $G = A \oplus B$, then every $f \in End(A)$ extends to an endomorphism of G by assigning the image of the second component of elements of G to 0.

Lemma 2.1 [2] If $G = B \oplus C$ and A is a fully invariant subgroup of G, then

$$A = (A \cap B) \oplus (A \cap C).$$

Definition 2.4 [4] Let $\{B_i : i \in I\}$ be a set of groups. An i^{th} tuple $f = (\cdots, b_i, \cdots)$ over this collection of groups B_i has exactly one coordinate b_i for each $i \in I$, that is, if f is a function defined over I, then $f(i) = b_i \in B_i$ for every $i \in I$. The set of all i^{th} tuples is called the ***direct product*** of the groups B_i, denoted by $\prod_{i\in I} B_i$.

Theorem 2.4 [2] Let A, B, A_i $(i \in I)$ and B_j $(j \in J)$ be abelian groups. Then

i. $Hom(\oplus_{i\in I} A_i, B) \cong \prod_{i\in I} (A_i, B)$

ii. $Hom(A, \prod_{j\in J} B_j) \cong \prod_{j\in J} Hom(A, B_j)$.

Remark 2.4.1 [4] If I is finite, then $\sum_{i\in I} A_i = \prod_{i\in I} A_i$.

Corollary 2.1 [4] Let $\{G_i : i \in I\}$ and $\{N_i : i \in I\}$ be families of groups such that $N_i \lhd G_i$ for each $i \in I$. Then $\prod_{i\in I} N_i \lhd \prod_{i\in I} G_i$ and $\prod_{i\in I} G_i / \prod_{i\in I} N_i \cong \prod_{i\in I} G_i / N_i$.

Theorem 2.5 [4] Let $\{f_i : G_i \to H_i\}$ be a family of homomorphisms of groups and let $f = \prod f_i$ be the map $\prod_{i\in I} G_i \to \prod_{i\in I} H_i$ given by $\{a_i\} \mapsto \{f_i(a_i)\}$. Then f is a homomorphism of groups such that $f\left(\prod_{i\in I} G_i\right) \subset \prod_{i\in I} H_i$, $\ker f = \prod_{i\in I} \ker f_i$ and $\operatorname{Im} f = \prod_{i\in I} \operatorname{Im} f_i$. Consequently, f is a monomorphism [respectively, epimorphism] if and only if each f_i is.

Definition 2.5 [2] An abelian group G is said to be ***torsion*** if every element of G has finite order.

Example 2.5.1 Let A be an abelian group and let $T = \{x \in A : |x| < \infty\}$. Then T is a subgroup of A called the torsion group of A.

Definition 2.6 [2] An abelian group G is ***torsion-free*** if every nonzero element of G has infinite order.

Example 2.6.1 The groups Z and Q are torsion-free since $nx = 0$, only if $x = 0$ for all $n \in Z^+$, $\forall x \in Z$ or $\forall x \in Q$.

Definition 2.7 [2] A group G is ***divisible*** if $G = nG$ for all $0 \neq n \in Z^+$.

Example 2.7.1 The groups $\{0\}$ and Q are divisible since $\{0\} = n\{0\}$ and $Q = nQ$ for all $n \in Z^+$.

Definition 2.8 [2] A subgroup M of a group G is said to be ***pure*** in G, denoted by $M \leq_* G$, if whenever $m = ny \in M$ for some $n \in Z^+$, $y \in G$, then there exists $m_1 \in M$ such that $m = nm_1$. Equivalently, M is pure in G if and only if $M \cap nG = nM$ for all $n \in Z^+$.

Example 2.8.1 Consider the group Z. Then $\{0\}$ and Z are the only pure subgroups of Z.

Proof: For all $n \in Z^+$, $n\{0\} = \{0\} = nZ \cap \{0\}$ and $nZ = nZ \cap Z$. Hence, $\{0\}$ and Z are pure subgroups of Z. Consider $mZ \leq Z$ for some $m \in Z^+$. Then for all $n \in Z^+$, $n(mZ) = nZ \cap mZ$ if $\gcd(n,m) = 1$. But for all $n, m \in Z^+$, $\gcd(n,m)$ is not always equal to 1. Thus $\{0\}$ and Z are the only pure subgroups of Z. ■

Example 2.8.2 The group of rational numbers Q has no proper nontrivial pure subgroups.

Proof: Let $\{0\} \neq H \leq_* Q$. Then $H \cap nQ = nH$ for all $n \in Z^+$. Since Q is divisible, $Q = nQ$ for all $n \in Z^+$. Hence, $H \cap nQ = H \cap Q = H$. Thus, $H = nH$, that is, H is divisible, by Definition 2.7. Let $\frac{a}{b} \in H$, $a, b \in Z - \{0\}$. Then $bH = H$ and so $a = b\left(\frac{a}{b}\right) \in H = aH$. Thus, there exists $h \in H$ such that $a = ah$ for some $h \in H$. Hence, $h = 1 \in H$. Since $H = nH, \frac{1}{n} = \left(\frac{1}{n}\right)n \in H = nH$ for all $0 \neq n \in Z$. This implies that $\frac{1}{n} \in H$ since H is torsion-free. Therefore, $H = Q$. ■

Theorem 2.6 [2] Let B, C be subgroups of A such that $C \leq B \leq A$. Then the following hold.

i. If C is pure in B and B is pure in A, then C is pure in A.
ii. If B is pure in A, then B/C is pure in A/C.
iii. If C is pure in A, B/C pure in A/C, then B is pure in A.

Definition 2.9 [2] A subgroup B of A is a ***direct summand*** of A if $A = B \oplus C$ for some $C \leq A$.

Theorem 2.7 [2] Every direct summand is a pure subgroup and in torsion-free groups, the intersection of pure subgroups is again pure.

Definition 2.10 [1] A group G is said to be ***irreducible*** if it has no proper nontrivial pure fully invariant subgroups.

Example 2.10.1 The groups Z and Q are irreducible.

Definition 2.11 [5] For any groups G and H, we say that

i. G is ***quasi-contained*** in H $\left(G \dot{\subseteq} H\right)$ if $nG \subseteq H$ for some $0 \neq n \in Z^+$;

ii. G is ***quasi-equal*** to H $\left(G \dot{=} H\right)$ if $G \dot{\subseteq} H$ and $H \dot{\subseteq} G$.

Definition 2.12 [5] A group G is said to be ***strongly irreducible*** if G is quasi-equal to each of its nontrivial fully invariant subgroups.

Remark 2.12.1 A strongly irreducible group is irreducible.
Proof: If *H* is a nontrivial fully invariant subgroup of a strongly irreducible group *G*, then $nG \subseteq H$ for some positive integer n. If in addition, $H \leq_* G$, then $nH = nG \cap H = nG$ for all $n \in Z^+$. Since *H* is torsion-free, $H = G$. Thus, a strongly irreducible group is irreducible. ■

The following concepts and results pertain to the kernel and kernel group of torsion-free abelian group.

Definition 2.13 [6] For any torsion-free abelian groups G and H, the ***kernel*** of H in G is

$$\ker(G,H) = \bigcap_{f \in Hom(H,G)} \ker f \ .$$

Example 2.13.1 Consider the group of integers Z and $\langle 0 \rangle \leq Z$. The kernel of $\langle 0 \rangle$ in Z is

$$\ker(Z,\langle 0 \rangle) = \bigcap_{f \in Hom(Z,\{0\})} = Z$$

since if $f \in Hom(Z,\{0\}),\ f(x) = 0$ for all $x \in Z$. Also, since $i_Z \in Hom(Z,\{0\})$ with $\ker i_Z = \{0\}$ and $\{0\} \subset \ker f$ for all $f \in Hom(Z,Z)$, the kernel of Z in Z is

$$\ker(Z,Z) = \bigcap_{f \in Hom(Z,Z)} = \{0\}.$$

Example 2.13.2 Given the group of rational numbers Q, observe that the kernel of $\{0\}$ in Q is

$$\ker(Q,\{0\}) = \bigcap_{f \in Hom(Q,\{0\})} = Q$$

since if $f \in Hom(Q,\{0\}),\ f(x) = 0$ for all $x \in Q$. Also, the kernel of Q in Q is

$$\ker(Q,Q) = \bigcap_{f \in Hom(Q,Q)} = \{0\}$$

since $i_Q \in Hom(Q,Q)$ with $\ker i_Q = \{0\}$ and $\{0\} \subset \ker f$ for all $f \in Hom(Q,Q)$.

The preceding examples can be generalized as follows.

Example 2.13.3 For any torsion-free abelian group G, the kernel of $\{0\}$ in G is

$$\begin{aligned}\ker(G,\{0\}) &= \bigcap_{f \in Hom(G,\{0\})} \ker f \\ &= G\end{aligned}$$

since for all $f \in Hom(G,\{0\}),\ f(g) = 0$ for all $g \in G$ and the kernel of G in G is

$$\ker(G,G) = \bigcap_{f \in Hom(G,G)} = \{0\}$$

since $i_G \in Hom(G,G)$ with $\ker i_G = \{0\}$ and $\{0\} \subset \ker f$ for all $f \in Hom(G,G)$.

Proposition 2.2 Let G and H be any torsion-free abelian groups. Then $\ker(G,H)$ is a pure fully invariant subgroup of G.

Proof: Let G and H be any torsion-free abelian groups. By Theorem 2.1 and Proposition 2.1, $\bigcap \ker f_i \leq G$, where $f_i \in Hom(G,H)$ for each i. Thus, by Definition 2.13,

$$Ker(G,H) = \bigcap \ker f_i \leq G,\ \ f_i \in Hom(G,H) \text{ for each } i.$$

Claim 1: $\ker(G,H) \leq_* G.$

Suppose that $ny \in \ker(G,H)$ with $y \in G$ for some $n \in Z^+$. Then $ny \in \ker f$ for all $f \in Hom(G,H)$. That is, for all $f \in Hom(G,H)$, $0 = f(ny) = nf(y)$. This implies that $f(y) = 0$, since $n \neq 0$ and H is torsion – free. So, $y \in \ker f$ for all $f \in Hom(G,H)$. Thus, $y \in \bigcap \ker f = \ker(G,H)$, $f \in Hom(G,H)$. Therefore, $\ker(G,H) \leq_* G$, by Definition 2.8.

Claim 2: $\ker(G,H)$ is fully invariant in G.

Let $\alpha \in End(G)$ and $y \in \ker(G,H) \leq G$. Then for all $f \in Hom(G,H)$, $f(y) = 0$ and $f\alpha \in Hom(G,H)$. Thus, $f(\alpha(y)) = f\alpha(y) = 0$ and hence $\alpha(y) \in \ker(G,H)$. ■

Definition 2.14 [6] A torsion-free abelian group G is a ***kernel group*** if $M = \ker(G, G/M)$ for every pure fully invariant subgroup M of G.

Example 2.14.1 The groups Z and Q are kernel groups since their only pure fully invariant subgroups are the trivial ones and itself with $\ker(Z, Z/\langle 0 \rangle) = \{0\}$, $\ker(Z, Z/Z) = Z$ and $\ker(Q, Q/\langle 0 \rangle) = \{0\}$, $\ker(Q, Q/Q) = Q$.

Lemma 2.2 Let $G = A \oplus B$ where $A, B \leq_* G$ and let M be a pure subgroup of G. Then $M \cap A$ and $M \cap B$ are pure in A and in B, respectively.

Proof: Clearly, $n(M \cap A) \subseteq nA \bigcap (M \cap A)$ for all $n \in Z^+$. Let $x \in nA \bigcap (M \cap A)$. Then, $x \in nA$ and $x \in M \cap A$. So, $x = na$ for some $a \in A$, $x \in M$, and $x \in A$. Since $x \in nG$, $x \in nG \bigcap M = nM$. Thus, $x = na \in nM$ or $a \in M$ since M is torsion-free. Hence, $a \in M \cap A$ and so $x = na \in n(M \cap A)$ which implies that $nA \bigcap (M \cap A) \subseteq n(M \cap A)$ for all $n \in Z^+$. Therefore, $n(M \cap A) = nA \bigcap (M \cap A)$ and $M \cap A \leq_* A$. Similarly, $M \cap B \leq_* B$. ■

3.0 Main Results

We first give and show that the irreducible and strongly irreducible groups are kernel groups.

Proposition 3.1 If G is an irreducible group, then G is a kernel group.

Proof: Suppose G is irreducible group. Then by Definition 2.10, $\{0\}$ and G are the only pure fully invariant subgroups of G. But $\ker(G, G/\{0\}) = \ker(G,G) = \{0\}$ and

$$\ker(G, G/G) = \ker(G, \{0\}) = G$$, by Example 2.13.3.

Therefore, G is a kernel group. ■

Proposition 3.2 A strongly irreducible group is a kernel group.

Proof: A strongly irreducible group G is irreducible, by Remark 2.12.1. Hence, by Proposition 3.1, G is a kernel group. ■

The following lemma considers the property of a kernel of torsion-free abelian group.

Lemma 3.1 For any torsion-free abelian groups *A*, *B*, *C* and *D* where $D < B$ and $C < A$,

$$\ker(A \oplus B, (A \oplus B)/(C \oplus D)) = \ker(A, A/C) \oplus \ker(B, B/D).$$

Proof: Let $x \in \ker(A \oplus B, (A \oplus B)/(C \oplus D))$. Then $x \in \ker f_i$ for each i,

$$f_i \in Hom(A \oplus B, (A \oplus B)/(C \oplus D)).$$

Hence, $f_i(x) = 0$ for each i. By Theorem 2.4 and Corollary 2.1,

$$Hom(A \oplus B, (A \oplus B)/(C \oplus D)) = Hom(A, (A \oplus B)/C \oplus D) \oplus Hom(B, (A \oplus B)/(C \oplus D))$$

$$= Hom(A, A/C) \oplus Hom(A, B/D) \oplus Hom(B, A/C) \oplus Hom(B, B/D).$$

Note that for $j = 1, 2$ if the map $f_i : G_j \to N_j$ given by $g_j \mapsto n_j$ is a homomorphism of groups then by Theorem 2.5, the map $\sum_{j=1}^{2} f_j : \sum_{j=1}^{2} G_j \to \sum_{j=1}^{2} N_j$ given by $g_1 + g_2 \mapsto n_1 + n_2$ is also a homomorphism of groups. Hence, the map $f_1 : A \to B/D$ is not defined since $A = G_1$ and $B/D = N_2$. Similarly the map $f_2 : B \to A/C$ is also not defined since $B = G_2$ and $A/C = N_1$. Thus, $Hom(A, B/D) = \{0\}$ and $Hom(B, A/C) = \{0\}$. And so,

$$Hom(A \oplus B, (A \oplus B)/(C \oplus D)) = Hom(A, A/C) \oplus Hom(B, B/D).$$

Hence, $f_i = f_{i_1} + f_{i_2}$ with $f_{i_1} \in Hom(A, A/C)$ and $f_{i_2} \in Hom(B, B/D)$. Thus, if $x = a + b$,

$$\begin{aligned} 0 &= f(x) \\ &= (f_{i_1} + f_{i_2})(a + b) \\ &= f_{i_1}(a) + f_{i_1}(b) + f_{i_2}(a) + f_{i_2}(b) \\ &= f_{i_1}(a) + 0 + 0 + f_{i_2}(b) \end{aligned}$$

$$= f_{i_1}(a) + f_{i_2}(b)$$

with $f_{i_1}(a) = 0$ for all i_1 and $f_{i_2}(b) = 0$ for all i_2. Hence, $a \in \ker f_{i_1}$ and $b \in \ker f_{i_2}$ for all i_1 and i_2. Thus,

$$x = a + b \in \bigcap \ker f_{i_1} \oplus \bigcap \ker f_{i_2}$$

$$= \ker(A, A/C) \oplus \ker(B, B/D).$$

Thus, $\ker(A \oplus B, (A \oplus B)/(C \oplus D)) \subseteq \ker(A, A/C) \oplus \ker(B, B/D)$. On the other hand, suppose $y \in \ker(A, A/C) \oplus \ker(B, B/D)$. Then $y = y_1 + y_2$, $y_1 \in \ker(A, A/C)$ and $y_2 \in \ker(B, B/D)$. Hence, $y_1 \in \ker f_{j_1}$, $f_{j_1} \in Hom(A, A/C)$ and $y_2 \in \ker f_{j_2}$, $f_{j_2} \in Hom(B, B/D)$. This implies that $f_{j_1}(y_1) = 0$ for all j_1 and $f_{j_2}(y_2) = 0$, for all j_2. Thus,

$$f_{j_1}(y_1 + y_2) + f_{j_2}(y_1 + y_2) = f_{j_1}(y_1) + f_{j_1}(y_2) + f_{j_2}(y_1) + f_{j_2}(y_2)$$

$$= (f_{j_1} + f_{j_2})(y_1) + (f_{j_1} + f_{j_2})(y_2)$$

$$= (f_{j_1} + f_{j_2})(y_1 + y_2)$$

with $f_{j_1} + f_{j_2} \in Hom(A \oplus B, (A \oplus B)/(C \oplus D))$ since $A/C, B/D \subset A \oplus B/C \oplus D$. And so, $f_{j_1}(y_1 + y_2) + f_{j_2}(y_1 + y_2) = f_j = 0$ for each j, $f_j \in Hom(A \oplus B, (A \oplus B)/(C \oplus D))$. Hence, $y \in \ker f_j$, that is, $y \in \ker(A \oplus B, (A \oplus B)/(C \oplus D))$ and

$$\ker(A, A/C) \oplus \ker(B, B/D) \subseteq \ker(A \oplus B, (A \oplus B)/(C \oplus D)).$$

Therefore, $\ker(A \oplus B, (A \oplus B)/(C \oplus D)) = \ker(A, A/C) \oplus \ker(B, B/D)$. ■

The next results characterize the kernel group with respect to its direct sum.

Theorem 3.1 Let *G* be a kernel group. If $G = H \oplus K$ and H is fully invariant in G, then H is a kernel group.

Proof: Suppose $G = H \oplus K$ such that G is a kernel group and H fully invariant in G. Let A be a pfi subgroup of H. Since $H \leq_* G$, by Theorem 2.6, A is pure in G. Also, A is fully invariant in H and H is fully invariant in G, by Theorem 2.2 A is fully invariant in G. So, by Definition 2.14 and Lemma 3.1,

$$A = \ker(G, G/A)$$

$$= \ker(H \oplus K, H \oplus K/A)$$

$$= \ker(H, H/A) \oplus \ker(K, K/A).$$

Since $A \leq H$ and $H \cap K = \{0\}$, then K/A is not defined. Thus, there are no nonzero maps from K to K/A, that is, $Hom(K, K/A) = \{0\}$. Thus, $A = \ker(H, H/A)$. Therefore, H is a kernel group. ■

Theorem 3.2 If $G = H \oplus K$ where H and K are kernel groups, then G is a kernel group.

Proof: Suppose $G = H \oplus K$ where H and K are kernel groups. Then by Theorem 2.7, $H, K \leq_* G$. Let A be a pure fully invariant subgroup of G. Then by Lemma 2.1, $A = (A \cap H) \oplus (A \cap K)$. By Theorem 2.7, $A \cap H$ and $A \cap K$ are pure in G. So, By Lemma 2.2, $A \cap H \leq_* H$ and $A \cap K \leq_* K$. Let $f \in End(H) \subset End(G)$ and $g \in End(K) \subset End(G)$ by Theorem 2.3. Then $f(A \cap H) \subseteq A \cap H$ and $g(A \cap K) \subseteq A \cap K$ since A is fully invariant in G. Thus, $A \cap H$ and $A \cap K$ are fully invariant in H and K, respectively. Since H and K are kernel groups, by Definition 2.14 and Lemma 3.1

$$A = A \cap H \oplus A \cap K$$

$$= \ker(H, H/(H \cap A)) \oplus \ker(K, K/(K \cap A))$$

$$= \ker(H \oplus K, (H \oplus K)/(H \cap A \oplus K \cap A)).$$

Therefore, it follows that G is a kernel group. ■

The following corollary shows that the direct sum of a collection of a finite number of kernel groups is again a kernel group.

Corollary 3.1 Let G be a torsion-free abelian group and $\{G_1, G_2, \cdots, G_m\}$ a collection of kernel groups. If $G = \oplus_{i=1}^{m} G_i$. then G is a kernel group.

Proof: Suppose $G = \oplus_{i=1}^{m} G_i$. Then we proceed by induction on m. If $m = 2$, then $G_1 \oplus G_2$ is a kernel group by Theorem 3.2. Suppose for $k > 2$, $\oplus_{i=1}^{k} G_i$ is a kernel group. Then we need to show that $\oplus_{i=1}^{k+1} G_i$ is a kernel group. Since $\oplus_{i=1}^{k+1} G_i = \left(\oplus_{i=1}^{k} G_i\right) \oplus G_{k+1}$ with $\oplus_{i=1}^{k} G_i$ and G_{k+1} as kernel groups, it follows from Theorem 3.2 that $\oplus_{i=1}^{k+1} G_i$ is a kernel group. Therefore, by Principle of Mathematical Induction, G is a kernel group. ■

The following corollary gives a characterization for the direct sum of collection of kernel groups.

Corollary 3.2 Let $G = \oplus G_i$, G_i fully invariant in G for all i. Then G is a kernel group if and only if G_i is a kernel group.

Proof: Suppose G is a kernel group. Since $G = \oplus G_i$ and each G_i is fully invariant in G, by Theorem 3.1, each G_i is a kernel group. Conversely, if each G_i is a kernel group then by Corollary 3.1, $\oplus G_i$ is a kernel group. Therefore, it follows that G is a kernel group. ■

4.0 Conclusions

In this paper, we have shown that irreducible and strongly irreducible groups are also a kernel groups. On the other hand, the concepts of kernel and kernels groups also hold on the direct sum of kernel and kernel groups of torsion-free abelian groups. In general, we have established that if *G* is a torsion-free abelian group and $\{G_1, G_2, \cdots, G_m\}$ a collection of kernel groups and if $G = \oplus_{i=1}^{m} G_i$, then *G* is a kernel group. Consequently, if $G = \oplus G_i$, where G_i fully invariant in a torsion-free abelian group G for all i, then G is a kernel group if and only if G_i is a kernel group.

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