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Deriving a Formula in Solving Reverse Fibonacci Means

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Abstract

Reverse Fibonacci sequence $\{J_n\}$ *is defined by the recurrence relation of* $J_n = 8(J_{n-1} - J_{n-2})$ *for* $n \geq 2$ *with* $J_0 = 0$ *and* $J_1 = 1$ *as initial terms. A few formulas have been derived for solving the missing terms of a sequence in books and mathematical journals, but not for the reverse Fibonacci sequence. Thus, this paper derived a formula that deductively solves the first missing term* $\{x_1\}$ *of the reverse Fibonacci sequence and is given by the equation*

$$x_1 = \frac{b + J_n 8a}{J_{n+1}}.$$

By using the derived formula for x_1*, it is now possible to solve the means of the reverse Fibonacci sequence as well as solving the sequence itself.*

Keywords: fibonacci sequence, reverse fibonacci sequence, Binet's formula, means

1.0 Introduction

Fibonacci sequence $\{F_n\}$ is a succession of numbers that can be obtained by adding the two preceding numbers and is defined by recurrence relation $F_n = F_{(n-1)} + F_{(n-2)}$ such that $F_0 = 0, F_1 = 1$ and $n \geq 2$ (Natividad, 2011a). In 2018, Janicko discovered a new sequence that is related to Fibonacci and he named it the "reverse Fibonacci sequence". This sequence is derived from the digital root of the Fibonacci sequence and is defined by the recurrence relation $J_n = 8(J_{(n-1)} - J_{(n-2)})$ for $n \geq 2$ with $J_0 = 0$ *and* $J_1 = 1$ as initial terms. The first 9 reverse Fibonacci numbers is given by the sequence

$1, 8, 56, 384, 2624, 17920, 122368, 835584, 5705728, \ldots$.

In 2019, Soucek and Janicko presented the geometric properties of the reverse Fibonacci sequence. Fibonacci and Fibonacci-like sequences have its application in finding the missing terms between the first and the last term given of a recurrence sequence (Natividad, 2011a; Natividad, 2012; Patan & Elizalde, 2017). The Binet-type formula of the reverse Fibonacci sequence is given by

$$J_n = \frac{(4 + 2\sqrt{2})^n - (4 - 2\sqrt{2})^n}{4\sqrt{2}}. \qquad (1)$$

Natividad (2011a) introduced a formula for finding the means of any Fibonacci-like sequences. In the same year, he also derived a formula for finding the means of a Pell sequence. In 2014, Sisodiya et al. presented a formula for finding the missing terms of a recurrence sequence using the Lucas sequence. Moreover, Patan and Elizalde (2017) developed an alternative method for finding the means of the Fibonacci and Fibonacci-like sequences.

This study aims to derive a formula for solving the missing terms of the reverse Fibonacci sequence between any two terms a and b of the sequence.

2.0 Results and Discussion

Definition 1. If $a, x_1, x_2, x_3, \ldots, x_{(n-1)}, x_n, b$ is a reverse Fibonacci sequence, then $x_1, x_2, x_3, \ldots, x_{(n-1)}, x_n$ are called reverse Fibonacci means between a and b.

Example 1. Consider the reverse Fibonacci sequence 8, x_1, x_2, x_3, 17920, ... Then, the terms x_1, x_2, and x_3 are called reverse Fibonacci means between 8 and 17920.

Derivation of the Formula for Solving the Reverse Fibonacci Means

In finding the means of any sequences, solving first for x_1 will make the other means easy to solve. This section will present the derivation of the formula for x_1 like the approach in Natividad (2011a, 2011b).

a. One mean, the sequence is a, x_1, b. Since

$$8(x_1 - a) = b,$$

then

$$x_1 = \frac{b + 8a}{8}.$$

b. Two means, the sequence is a, x_1, x_2, b. Since

$$8(x_1 - a) = x_2$$

and

$$8(x_2 - x_1) = b,$$

then

$$x_1 = \frac{b + 64a}{56}.$$

c. Three means, the sequence is a, x_1, x_2, x_3, b. Since

$$8(x_1 - a) = x_2$$

$$8(x_2 - x_1) = x_3$$

and

$$8(x_2 - x_3) = b,$$

then

$$x_1 = \frac{b + 448a}{384}.$$

d. Four means, the reverse Fibonacci sequence is a, x_1, x_2, x_3, x_4, b. Since

$$8(x_1 - a) = x_2,$$

$$8(x_2 - x_1) = x_3,$$

$$8(x_3 - x_2) = x_4,$$

and

$$8(x_4 - x_3) = b,$$

then

$$x_1 = \frac{b + 3072a}{2624}.$$

We can observe in the formula for x_1, the numerical coefficient of a in the numerator is following the reverse Fibonacci sequence multiplied by 8. Also, the denominator of the formulas are following the reverse Fibonacci sequence. The formula can be exemplified as

$$x_1 = \frac{b + J_n 8a}{J_{n+1}} \quad (2)$$

Table 1. *Relationship of Number of Missing Terms with Numerator and Denominator Formulas*

Number of Missing Term	Coefficient of a in Numerator	Coefficient of Denominator
1	8(1)	8
2	8(8)	56
3	8(56)	384
4	8(384)	2624
.	.	.
.	.	.
.	.	.
n	$8\left[\frac{(4+2\sqrt{2})^n - (4-2\sqrt{2})^n}{4\sqrt{2}}\right]$	$\frac{(4+2\sqrt{2})^{n+1} - (4-2\sqrt{2})^{n+1}}{4\sqrt{2}}$

Theorem 1. *For any natural number* $n\geq 1$*, the coefficient of* a *is given by*

$$C_n = 8\left[\frac{(4+2\sqrt{2})^n-(4-2\sqrt{2})^n}{4\sqrt{2}}\right]$$

where n is the number of missing terms and C_n *is the* n^{th} *term of the coefficient in* a*.*

Proof. The theorem will be proved using the Principle of Mathematical Induction. Let *P(n)* be

$$C_n = 8\left[\frac{(4+2\sqrt{2})^n-(4-2\sqrt{2})^n}{4\sqrt{2}}\right], \text{ for all } n\geq 1$$

When $n = 1$.

$$C_1 = 8\left[\frac{(4+2\sqrt{2})^1-(4-2\sqrt{2})^1}{4\sqrt{2}}\right]$$
$$= 8(1)$$
$$= 8$$

which is true from Table 1. So, $P(1)$ is true. Now, suppose that $P(n)$ is true for $n = 1, 2, \ldots, k$, i.e.,

$$C_k = 8\left[\frac{(4+2\sqrt{2})^k-(4-2\sqrt{2})^k}{4\sqrt{2}}\right].$$

We need to show that $P(n)$ is true for $n = k + 1$, i.e.,

$$C_{k+1} = 8\left[\frac{(4+2\sqrt{2})^{k+1}-(4-2\sqrt{2})^{k+1}}{4\sqrt{2}}\right].$$

To do this, we shall subtract C_{k-1} and multiply 8 to both sides of $P(k)$, i.e.,

$$8(C_k - C_{k-1}) = 8\left(8\left[\frac{(4+2\sqrt{2})^k-(4-2\sqrt{2})^k}{4\sqrt{2}}\right] - 8\left[\frac{(4+2\sqrt{2})^{k-1}-(4-2\sqrt{2})^{k-1}}{4\sqrt{2}}\right]\right)$$

$$\Rightarrow C_{k+1} = 8\left(8\left(\frac{(4+2\sqrt{2})^k-(4-2\sqrt{2})^k}{4\sqrt{2}} - \frac{(4+2\sqrt{2})^{k-1}-(4-2\sqrt{2})^{k-1}}{4\sqrt{2}}\right)\right)$$

$$= 8\left(8\left(\frac{(4+2\sqrt{2})^{k+1}((4+2\sqrt{2})^{-1}-(4+2\sqrt{2})^{-2})-(4-2\sqrt{2})^{k+1}((4-2\sqrt{2})^{-1}-(4-2\sqrt{2})^{-2})}{4\sqrt{2}}\right)\right)$$

$$= 8\left(8\left(\frac{1}{8}\cdot\frac{(4+2\sqrt{2})^{k+1}-(4-2\sqrt{2})^{k+1}}{4\sqrt{2}}\right)\right)$$

$$= 8\left[\frac{(4+2\sqrt{2})^{k+1}-(4-2\sqrt{2})^{k+1}}{4\sqrt{2}}\right].$$

Thus, by PMI, the theorem is verified. □

By replacing the formula in (1) to the formula in (2), we obtain the new formula for x_1 that is

$$x_1 = \frac{b + \left[\frac{(4+2\sqrt{2})^n-(4-2\sqrt{2})^n}{4\sqrt{2}}\right]8a}{\frac{(4+2\sqrt{2})^{n+1}-(4-2\sqrt{2})^{n+1}}{4\sqrt{2}}}$$

$$= \frac{4b\sqrt{2}+[(4+2\sqrt{2})^n-(4-2\sqrt{2})^n]8a}{(4+2\sqrt{2})^{n+1}-(4-2\sqrt{2})^{n+1}} \qquad (3)$$

where x_1 is the first mean in reverse Fibonacci sequence, a is the first term given, b is the last term given, and n is the number of means.

Example 2. Insert 4 reverse Fibonacci means between 1 and 17920.

Solution:

Given $a = 1, b = 17920$ and $n = 4$. Then using the formula in (3),

$$x_1 = \frac{4(17920)\sqrt{2} + \left[(4+2\sqrt{2})^4 - (4-2\sqrt{2})^4\right]8(1)}{(4+2\sqrt{2})^5 - (4-2\sqrt{2})^5}$$

Using scientific calculator, we get

$$x_1 = 8.$$

Since x_1 was already solved, then we can find x_2 by

$$x_2 = 8(x_1 - a)$$
$$= 8(8-1)$$
$$= 8(7)$$
$$= 56.$$

The same way for solving x_3 and x_4. Thus, the sequence is 1, 8, 56, 384, 2624, 17920, ...

Proof of the Formula for Solving Reverse Fibonacci Means

The Binet's formula for the second-order linear recurrence sequence $\{W_n\}$ given by equation

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \qquad (4)$$

where $A = c - a\beta$, $B = c - a\alpha$ and, a and c are the initial terms of the sequence, will be used to prove the following theorem (Rabago, 2012).

Theorem 2. *If a and b are any two terms of the reverse Fibonacci sequence, then the first missing* term between a and b is given by

$$x_1 = \frac{b + J_n 8a}{J_{n+1}}, \qquad (5)$$

where n is *the number of missing terms and J_n is the n^{th} term of the reverse Fibonacci sequence. Moreover,*

$$x_1 = \frac{4b\sqrt{2} + \left[\left(4+2\sqrt{2}\right)^n - \left(4-2\sqrt{2}\right)^n\right]8a}{\left(4+2\sqrt{2}\right)^{n+1} - \left(4-2\sqrt{2}\right)^{n+1}}. \qquad (6)$$

Proof. Let $W_0 = a$ be the first term of the sequence and b be the last term of the sequence. If n is the number of missing terms between a and b then, $b = W_{n+1}$. Now suppose that x_1 is the second term in the sequence, hence the Binet's formula for the sequence is given by

$$W_n = \frac{(x_1 - a\beta)\alpha^n - (x_1 - a\alpha)\beta^n}{4\sqrt{2}},$$

where $\alpha = 4 + 2\sqrt{2}$ and $\beta = 4 - 2\sqrt{2}$. This would imply that,

$$\begin{aligned} b = W_{n+1} &= \frac{(x_1 - a\beta)\alpha^{n+1} - (x_1 - a\alpha)\beta^{n+1}}{4\sqrt{2}} \\ &= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^{n+1}\beta - \alpha\beta^{n+1})a}{4\sqrt{2}} \\ &= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^n - \beta^n)a\alpha\beta}{4\sqrt{2}} \\ &= J_{n+1}x_1 - J_n 8a. \end{aligned}$$

This proves equation (5).

Using (1), we can further write (5) as follows:

$$x_1 = \frac{b + \left[\frac{(4+2\sqrt{2})^n - (4-2\sqrt{2})^n}{4\sqrt{2}}\right]8a}{\frac{(4+2\sqrt{2})^{n+1} - (4-2\sqrt{2})^{n+1}}{4\sqrt{2}}}$$

$$= \frac{4b\sqrt{2} + [(4+2\sqrt{2})^n - (4-2\sqrt{2})^n]8a}{(4+2\sqrt{2})^{n+1} - (4-2\sqrt{2})^{n+1}},$$

which is exactly (6). □

3.0 Conclusion

The reverse Fibonacci sequence has been constructed from the digital root of the Fibonacci sequence. Means of the reverse Fibonacci sequence are between any two terms a and b of the sequence. A formula was derived to solve the means of the reverse Fibonacci sequence given its first and last terms. Moreover, it is now possible to solve the means of the reverse Fibonacci sequence using the formula for x_1. The students can now use the formula without difficulty since it needs basic algebra only. For future research work, one may try to determine the number of missing terms of the reverse Fibonacci sequence between any of their two terms a and b.

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